Department of Mathematical and Computational Sciences, NITK, Surathkal
Linear Algebra and Matrices (MA 904)
https://sam.nitk.ac.in/sites/default/MA904-ps-5.pdf

Instructor : P. Sam Johnson

Problem Sheet 5

1. Given W_1, W_2 vector subspaces of V, does there exist any smallest vector subspace W_3 containing W_1 and W_2?

2. Let W be a vector subspace of V. What is $w + W$ if $w \in W$? What is $W + W$? Is is true that $w + W = W$ if and only if $w \in W$?

3. What is the dimension of the set of reals $\mathbb{R}$ over $\mathbb{Q}$?

4. Say true or false: If x and y are linearly independent vectors in V, then so are $x + y$ and $x - y$.

5. Let $T : V \to W$ be linear. Let V^* and W^* be their duals. We define a map $T^* : W^* \to V^*$ as follows. Given $g \in W^*$, $Tg \in V^*$ is given by $Tg(v) := g(Tv)$ for all $v \in V$. Show that T^* is a linear map. It is called the *adjoint* of T.

6. True or false? If V, W are vector spaces and $T : V \to W$ is a linear map and $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in V, then $\{T(v_i)\}_{i=1}^n$ is linearly independent.

7. Let $V = \mathbb{R}^n$ and A be a $n \times n$ matrix. If $Ax = 0$ has a unique solution then $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

8. Can you construct a linear map $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that $\mathtt{Im}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$?

9. Can you construct a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $\mathtt{Im}(T) = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$?

10. Check whether the following are inner products or not.
    (a) $\langle z, w\rangle := \mathtt{Re}(z\overline{w})$ on $\mathbb{C}$.
    (b) $\langle (x_1, x_2), (y_1, y_2)\rangle := y_1(2x_1 + x_2) + y_2(x_1 + x_2)$ on $\mathbb{R}^2$.
    (c) $\langle A, B\rangle := \mathtt{tr}AB^t$ on $\mathbb{M}(n, \mathbb{R})$. The trace of the matrix A, $\mathtt{tr}(A)$ is the sum of diagonals.

11. Prove that two nonzero vectors orthogonal to each other are linearly independent.

12. Say true or false. Zero is the only vector is orthogonal to every vector of V.

13. Let W_i, $i = 1, 2$, be vector subspaces of V. Assume that each vector is one of them is orthogonal to all of the other. Then prove that $W_1 \cap W_2 = \{0\}$.

14. True or False : The following subsets of $\mathbb{R}$ form subspaces of $\mathbb{R}$ over $\mathbb{Q}$:
    (a) $\mathbb{Q}$, and
    (b) $\{\alpha + \beta\sqrt{2} + \gamma\sqrt{3} : \alpha, \beta, \gamma \in \mathbb{Q}\}$.

    If the above subsets are subspaces, find basis.

15. What is the matrix representation of the conjugation map $z \mapsto \overline{z}$ of $\mathbb{C}$ with respect to the "usual" basis of $\mathbb{C}$?