Linear Transformations

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Overview

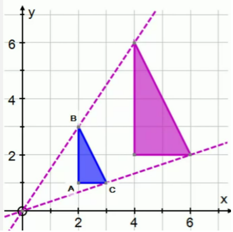
Let A be an $m \times n$ matrix and x be an n-dimensional vector.

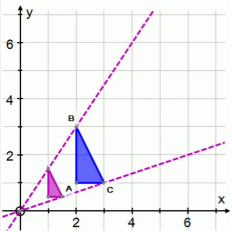
When A multiplies x, we can think of it as **transforming** that vector into a new vector Ax. This happens at every point x of the n-dimensional space $\mathbb{R}^n$.

The whole space is transformed, or "mapped into," by the matrix A.

We discuss transformtion of this kind, in general vector space settings.

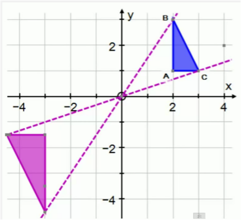
We start with four examples of the transformations that come from matrices. A multiple of the identity matrix, $A = cI$, **streches** every vector by the same factor c. The whole space expands or contracts (or somehow goes through the origin and out the opposite side, when c is negative).

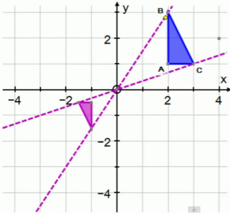




Positive Scaling

Enlargement / Shrink Scalar Factor $k > 0$**, Centre** $(0, 0)$

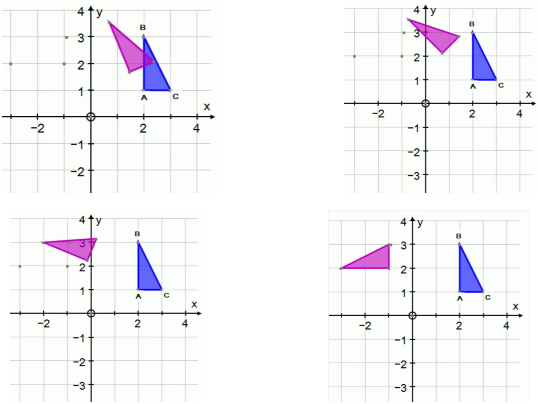




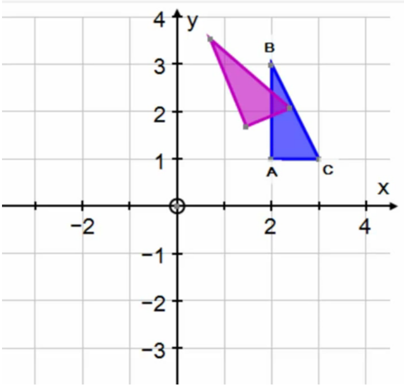
Negative Scaling

Enlargement / Shrink Scalar Factor $k < 0$**, Centre** $(0, 0)$

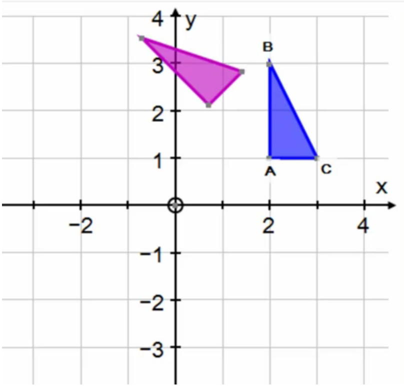
A **rotation** matrix turns the whole space around the origin. The following example turns all vectors in the triangle with vertices $A(2,1)$, $B(2,3)$ and $C(3,1)$ through 90°.



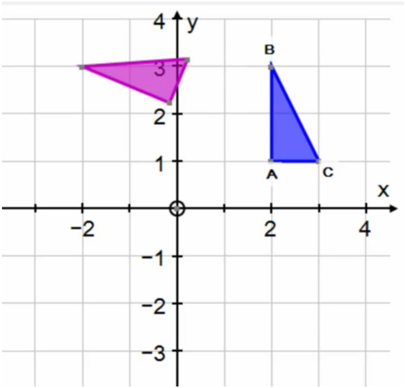
Rotation by 270° (step by step) : Figure 1 (step 1)



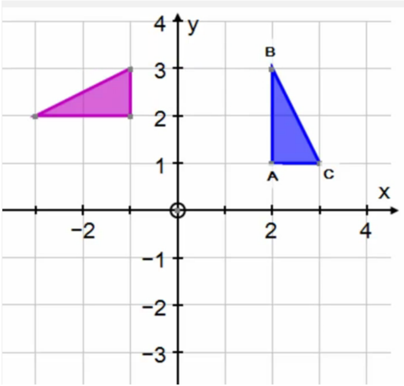
Rotation by 270° (step by step) : Figure 2 (step 2)



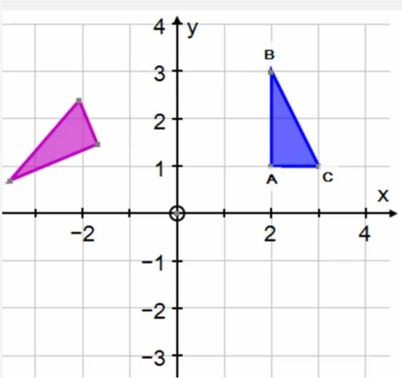
Rotation by 270° (step by step) : Figure 3 (step 3)



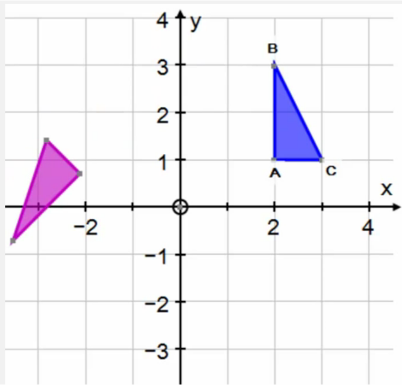
Rotation by 270° (step by step) : Figure 4 (step 4)



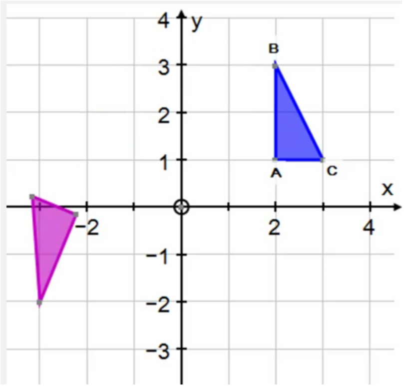
Rotation by 270° (step by step) : Figure 5 (step 5)

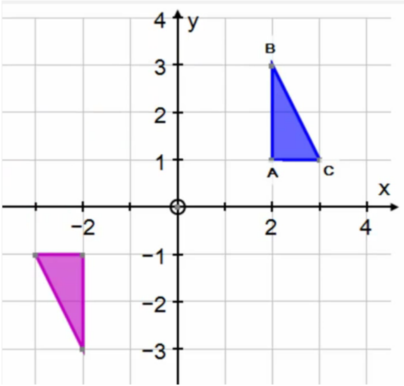


Rotation by 270° (step by step) : Figure 6 (step 6)



Rotation by 270° (step by step) : Figure 7 (step 7)




y
x
B
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3
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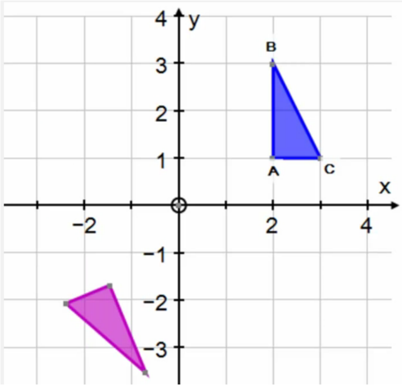
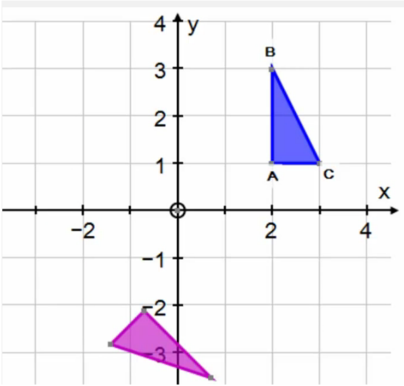
−2
−3
−2
2
4

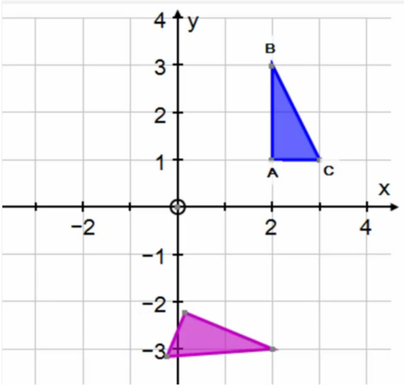
Rotation by 270° (step by step) : Figure 9 (step 9)



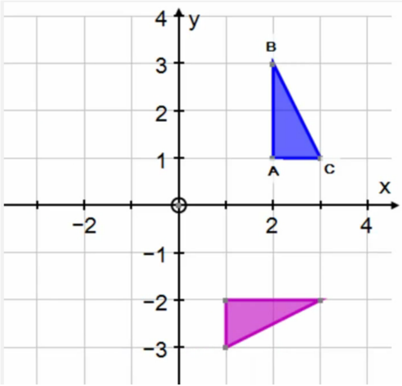
Rotation by 270° (step by step) : Figure 10 (step 10)



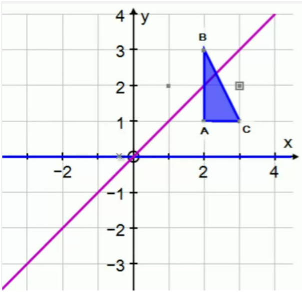
Rotation by 270° (step by step) : Figure 11 (step 11)

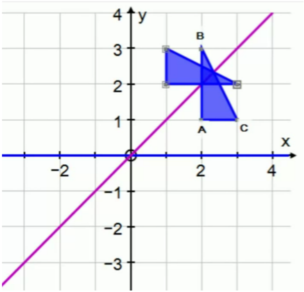


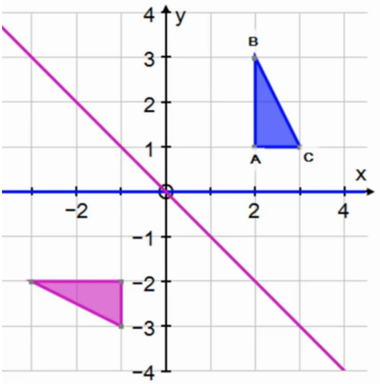
Rotation by 270° (step by step) : Figure 12 (last step)



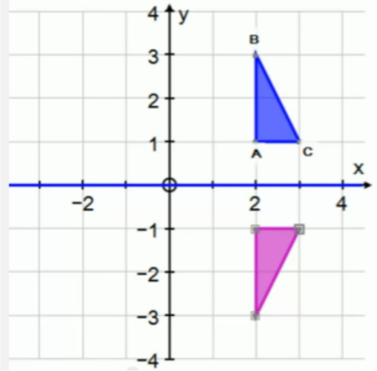
A **reflection** matrix transforms every vector into its image on the opposite side of a mirror. In this example the mirror is the 45° line $y = x$, and a point $(2, 2)$ is unchanged. A point like $(2, -2)$ is reversed to $(-2, 2)$. On a combination like $(2, 2) + (2, -2) = (4, 0)$, the matrix leaves one part and reverses the other part. The reflection matrix is also a permutation matrix! It is algebraically so simple, sending (x, y) to (y, x).



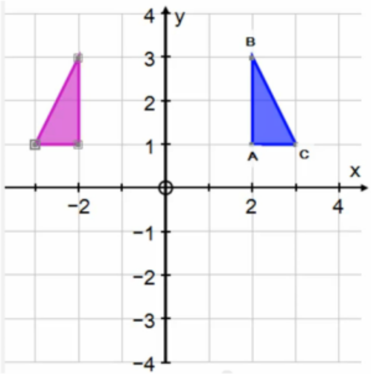




Reflection about the line $y = -x$



Reflection about the x-axis



Reflection about the y-axis

Projection

The fourth example is simple in both respects : A **projection** matrix takes the whole space onto a lower-dimensional subspace (and therefore fails to be invertible).

The example transforms each vector (x, y) in the plane to the nearest point $(x, 0)$ on the horizontal axis. That axis is the column space of A, and the vertical axis (which projects onto the origin) is the nullspace.

Everything is not possible with matrices !

It is also important to recognize that matrices cannot do everything, and some transformations are not possible with matrices:

1. It is impossible to move the origin, since $A0 = 0$ for every matrix.
2. If the vector x goes to x', then $2x$ must go to $2x'$. In general cx must go to cx', since $A(cx) = cA(x)$.
3. If the vectors x and y go to x' and y', then their sum $x + y$ must go to $x' + y'$, since $A(x + y) = Ax + Ay$.

Motivation for defining linear transformation

Transformations that obey the above three rules are called **linear transformations**. Those conditions can be combined into a single requirement: For all numbers c and d and all vector x and y, matrix multiplication satisfies the rule of linearity: $A(cx + dy) + c(Ax) + d(Ay)$. Every transformation that meets this requirement is a linear transformation.

Any matrix leads immediately to a linear transformation. The more interesting question is in the opposite direction : Does every linear transformation lead to a matrix? It is now objective to answer that question (affirmatively, in n dimensions). This theory is the foundation of an approach to linear algebra.

Linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$

A tranformation need not go from $\mathbb{R}^n$ to the same space $\mathbb{R}^n$. It is absolutely permitted to transform vectors in $\mathbb{R}^n$ to vectors in a different space $\mathbb{R}^m$. That is exactly what is done by m by n matrix!

The original vector x has n components, and the tranformed vector Ax has m components. The rule of linearity is equally satisfied by rectangular matrices, so they also product linear transformations.

Differentiation

The operation of **differentiation**, $A = d/dt$, is linear:

$$Ap = \frac{d}{dt}(a_0 + a_1 t + \cdots + a_n t^n) = a_1 + \cdots + n a_n t^{n-1}.$$

Its nullspace is the one-dimensional space of constant polynomials: $da_0/dt = 0$.

Its column space is the n-dimensional space P_{n-1}; the right side of above is always in that space.

The sum of nullity $(= 1)$ and rank $(= n)$ is the dimension of the original space P_n.

Integration

Integration from 0 to t is also linear (it takes P_n to P_{n+1}):

$$Ap = \int_0^t (a_0 + a_1 t + \cdots + a_n t^n) dt = a_1 + \cdots + \frac{a_n}{n+1} t^{n+1}.$$

This time there is no nullspace (except for the zero vector, as always!) but integration does not produce all polynomials in P_{n+1}.

The right side of above has no constant term. Probably the constant polynomials will be the left nullspace.

Multiplication by a fixed polynomial

Multiplication by a fixed polynomial like $2 + 3t$ is linear:

$$Ap = (2 + 3t)(a_0 + a_1 t + \cdots + a_n t^n) = 2a_0 + a_1 t + \cdots + 3a_n t^{n+1}.$$

Again this transforms P_n to P_{n+1}, with no nullspace except $p = 0$.

Of course, most transformations are not linear - for example to square the polynomial ($Ap = p^2$), or to add 1 ($Ap = p + 1$), or to keep the positive coefficients ($A(t - t^2) = t$).

It will be linear transformations, and only those, that lead us back to matrices.

Transformation represented by matrices

If we know Ax for each vector in a basis, then we know Ax for each vector in the entire space.

Next we try a new problem – to find a matrix that represents differentiation, and a matrix that represents integration.

That can be done as soon as we decide on a basis. For the polynomials of degree 3 (the space P_3 whose dimension is 4) there is a natural choice for the four basis vectors: $p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3$.

Transformation represented by matrices

That basis is not unique (it never is), but some choice is necessary and this is the most convenient. We look to see what differentiation does to those four basis vectors.

Their derivatives are $0, 1, 2t, 3t^2$, or
$Ap_1 = 0, Ap_2 = p_1, Ap_3 = 2p_2, Ap_4 = 3p_3$.

Then the matrix corresponding to it would be

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the "differentiation matrix".

Transformation represented by matrices

The derivative of any other combination like $p = 2 + t - t^2 - t^3$ is decided by linearity, and there is nothing new about that - it is the only way to differentiate. The matrix can differentiate that polynomial:

$$\frac{dp}{dt} = Ap \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \end{bmatrix} \rightarrow 1 - 2t - 3t^2.$$

In short, the matrix carries all the essential information. If the basis is known, and the matrix is known, then the linear transformation is known.

Transformation represented by matrices

For transformation from a space to itself one basis is enough. A transformation from one space to another requires a basis for each.

Suppose the vectors $x_1, \ldots, x_n$ are a basis for the space V, and $y_1, \ldots, y_m$ are basis for W. Then each linear transformation A from V to W is represented by a matrix.

The jth column is found by applying A to the jth basis vector; the result Ax_j is a combination of the y's and the coefficients in that combination go into column j:

$$Ax_j = a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m.$$

Transformation represented by matrices

We do the same for integration. That goes from cubics to quartics, transforming $V = P_3$ into $W = P_4$, so for W we need a basis.

The natural choice is $y_1 = 1, y_2 = t, y_3 = t^2, y_4 = t^3, y_5 = t^4$.

The matrix will be 5 by 4.

$$A_{\text{int}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

This is the "integration matrix".

Differentiation and integration as inverse operations

We think of differentiation and integration as inverse operations. Or at least integration followed by differentiation leads back to the original function. To make that happen for matrices, we need the differentiation matrix from quartics down to cubics, which is 4 by 5:

$$A_{\text{diff}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \text{ and } A_{\text{diff}} A_{\text{int}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Differentiation is a **left-inverse** of integration. But rectangular matrices cannot have two-sided inverses! In the opposite order, it cannot be true that $A_{\text{int}} A_{\text{diff}} = I$.

Rotations Q, Projections P, and Reflections H

We began with 90° rotations onto the x-axis, and reflections through the 45° line. Their matrices were especially simple:

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{(rotation)}, P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{(projection)}, H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{(reflection)}.$$

Of course the underlying linear transformations of the xy-plane are also simple. But it seems that rotations through other angles, and projections onto other lines, and reflections in other mirrors, are almost as easy to visualize. They are still linear transformations, provided the origin is fixed: $A0 = 0$.

They must be represented by matrices. Using the natural basis $(1, 0)$ and $(0, 1)$, we want to discover what those matrices are.

Rotation through an angle θ

The first vector $(1,0)$ goes to $(\cos\theta, \sin\theta)$, whose length is still one; it lies on the "θ-line".

The second basis vector $(0,1)$ rotates into $(-\sin\theta, \cos\theta)$. Those vectors go into the columns of the matrix $Q_\theta = P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

This family of rotations Q_θ is a prefect chance to test the correspondence between transformations and matrices:

1. Does the inverse of Q_θ equals $Q_{-\theta}$ (rotation backward through θ)? Yes.
2. Does the square of Q_θ equal $Q_{2\theta}$ (rotation through a double angle)? Yes.
3. What is the product of Q_θ and Q_ϕ (rotation through θ and ϕ)?

Linear Transformation

We now discuss transformtion of this kind, in general vector space settings.

Definition 1.

Let V and W be vector spaces over the same field F. A **linear transformation from V into W** *is a function T from V into W such that*

$$T(c\alpha + \beta) = c(T\alpha) + T\beta$$

for all α and β in V and all scalars c in F.

Example 2.

If V is any vector space, the **identity transformation** *I defined by $I\alpha = \alpha$, is a linear transformation from V into V.*

The **zero transformation** *0, defined by $0\alpha = 0$, is a linear transformation from V into V.*

Example 3.

Let F be a field and let V be the space of polynomial functions f from F into F, given by $f(x) = c_0 + c_1x + \cdots + c_kx^k$. Let

$$(Df)(x) = c_1 + 2c_2x + \cdots + kc_kx^{k-1}.$$

Then D is a linear transformation from V into V, called the **differentiation transformation**.

Example 4.

Let A be a fixed $m \times n$ matrix with entries in the field F. The function T defined by $T(X) = AX$ (where X is in $F^{n\times 1}$) is a linear transformation from $F^{n\times 1}$ into $F^{m\times 1}$.

The function U defined by $U(\alpha) = \alpha A$ (where α is an m-tuple) is a linear transformation from F^m into F^n.

Example 5.

Let P be a fixed $m \times m$ matrix with entries in the field F and Q be a fixed $n \times n$ matrix over F. Define a function T from the space $F^{m \times n}$ into itself by

$$T(A) = PAQ.$$

Then T is a linear transformation from $F^{m \times n}$ into $F^{m \times n}$.

Example 6 (Integration Transformation).

Let $\mathbb{R}$ be the field of real numbers and let V be the space of all functions from $\mathbb{R}$ and $\mathbb{R}$ which are **continuous**. *Define T by*

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

Then T is a linear transformation from V into V. The function Tf is not only continuous but has a continuous first derivative.

Linear Transformation

Linear transformation preserves linear combinations. That is, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are vectors in V, and $c_1, c_2, \ldots, c_n$ are scalars, then

$$T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n) = c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_n T(\alpha_n).$$

In a calculus course, one would probably call a function linear if its graph is a straight line.

Suppose V is the vector space $\mathbb{R}$. A linear transformation from V into V is then a particular type of real-valued function on the real line $\mathbb{R}$.

A linear transformation from $\mathbb{R}$ into $\mathbb{R}$, will be a function from $\mathbb{R}$ into $\mathbb{R}$, the graph of which is a straight line **passing through the origin.**

If T is a linear transformation from V into W, then

(a) $T(0) = 0$.

(b) the **range space** of T, denoted by $R(T)$, is a subspace of W.

$$R(T) = \Big\{\beta \in W : \beta = T\alpha, \ \text{ for some } \ \alpha \in V\Big\}.$$

(c) the set of all vectors α in V such that $T\alpha = 0$, is called the **null space** of T, denoted by $N(T)$.

$$N(T) = \Big\{\alpha \in V : T\alpha = 0\Big\}$$

is another interesting subspace associated with the linear transformations T.

Theorem 7.

Let V be a finite-dimensional vector spce over the field F and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V.

Let W be a vector space over the same field F and let $\beta_1, \beta_2, \ldots, \beta_n$ by any vectors (not necessarily distinct) in W.

Then there is precisely **one linear transformation** *T from V into W such that*

$$T\alpha_j = \beta_j, \quad j = 1, 2, \ldots, n.$$

If V and W are non-zero vector spaces, there are many functions from V into W. Theorem (7) helps to underscore the fact that the **functions which are linear are extremely useful**.

Let T be a linear transformation from the $m-$tuple space F^m into the n-tuple space F^n. Theorem (7) tells us that T is uniquely determined by the sequence of vectors $\beta_1, \beta_2, \ldots, \beta_m$ where

$$\beta_i = Te_i \quad i = 1, 2, \ldots, m.$$

In short, T is uniquely determined by the images of the standard basis vectors.

The determination is

$$\begin{aligned} \alpha &= (x_1, x_2, \ldots, x_m) \\ T\alpha &= x_1\beta_1 + x_2\beta_2 + \cdots + x_m\beta_m \end{aligned}$$

If B is the $m \times n$ matrix which has row vectors $\beta_1, \beta_2, \ldots, \beta_m$, this says that

$$T\alpha = \alpha B.$$

In other words, if $\beta_i = (B_{i1}, B_{i2}, \ldots, B_{in})$, then

$$T(x_1, x_2, \ldots, x_n) = [x_1, x_2, \ldots, x_m] \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}.$$

This is a very explicit description of the linear transformation.

Exercise 8.

Find the unique linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(1,2) = (3,2,1) \quad \textit{and} \quad T(3,4) = (6,5,4).$$

Remarks :

- We shall not pursue the particular description $T\alpha = \alpha B$ because it has the matrix B on the **right of the vector** α, and that can lead to some confusion.
- The example helps to show that one can give an explicit and reasonably simple description of all linear transformations from F^m into F^n.

Definition 9.

Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If V is finite-dimensional, the **rank** *of T is the dimension of $R(T)$, and the* **nullity** *of T is the dimension of $N(T)$.*

The following is one of the most important results in linear algebra, called **rank-nullity theorem**.

Theorem 10 (Rank-Nullity Theorem).

Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite-dimensional. Then

$$rank\ (T) + nullity\ (T) = dim\ V.$$

Row and Column Ranks of a Matrix

Theorem 11.

If A is an $m \times n$ matrix with entries in the field F, then

$$\textit{row rank } (A) = \textit{column rank } (A).$$

The proof of Theorem (11) depends upon explicit calculations concerning systems of linear equations.

There is a more conceptual proof which does not rely on such calculations. We shall discuss such a proof later.

Exercises 12.

1. *Which of the following functions T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations?*
   (i) $T(x_1, x_2) = (1 + x_1, x_2)$;
   (ii) $T(x_1, x_2) = (x_2, x_1)$;
   (iii) $T(x_1, x_2) = (x_1^2, x_2)$;
   (iv) $T(x_1, x_2) = (\sin x_1, x_2)$;
   (v) $T(x_1, x_2) = (x_1 - x_2, 0)$.
2. *Find the range, rank, null space, and nullity for the zero transformation and the identity transformation on a finite-dimensional space V.*
3. *Describe the range and the null spaces for the differentiation transformation (discussed earlier). Do the same for the integration transformation.*

Exercises 13.

1. *Suppose we have the following*

$$\alpha_1 = (1, -1), \quad \beta_1 = (1, 0)$$
$$\alpha_2 = (2, -1), \quad \beta_2 = (0, 1)$$
$$\alpha_3 = (-3, 2), \quad \beta_3 = (1, 1).$$

   Is there a linear transformation T from $\mathbb{R}^2$ and $\mathbb{R}^2$ such that $T\alpha_1 = \beta_i$ for $i = 1, 2$ and 3?

2. *Describe explicitly the linear transformation T from F^2 into F^2 such that $Te_1 = (a, b)$, $Te_2 = (c, d)$.*
3. *Describle explicitly a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has as its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.*

Exercises 14.

1. *Let F be a subfield of the complex numbers and let T be the function from F^2 into F^2 defined by*

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

   (i) *Verify that T is a linear transformation.*
   (ii) *If (a, b, c) is a vector in F^2, what are the conditions on $a, b,$ and c that the vector be in the range of T? What is the rank of T?*
   (iii) *What are the conditions on $a, b,$ and c that (a, b, c) be in null space of T? What is the nullity of T?*

2. *Let V be the vector space of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix. If*

$$T(A) = AB - BA$$

verify that T is a linear transformation from V into V.

Exercises 15.

1. *Let V be the set of all complex numbers regarded as a vector space over the field of real numbers (usual operations). Find a function from V into V which is a linear transformation on the above vector space, but which is not a linear transformation on $\mathbb{C}$, i.e., which is not complex linear.*
2. *Let V be the space of $n \times 1$ matrices over F and let W be the space of $m \times 1$ matrices over F. Let A be a fixed $m \times n$ matrix over F and let T be the linear transformation from V into W defined by*

$$T(X) = AX.$$

Prove that T is the zero transformation if and only if A is the zero matrix.

Exercises 16.

1. *Leet V be an n-dimensional vector space over the field F and let T be a linear transformation from V into V such that the range and null space of T are identical. Prove tht n is even. (Can you give an example of such a linear transformation T?)*
2. *Let V be a vector space and T a linear transformation from V into V. Prove the the following two statements about T are equivalent.*
   (i) *The intersection of the range of T and the null space of T is the zero subspace of V.*
   (ii) *If $T(T\alpha) = 0$, then $T\alpha = 0$.*

The Algebra of Linear Transformations

The set of linear transformations from V into W, inherits a natural vector space structure.

The set of linear transformations **from a space** V **into itself** has even more algebraic structure, because ordinary **composition of functions** provides a "**multiplication**" of such transformations.

The Algebra of Linear Transformations

Theorem 17.

Let V and W be vector spaces over the field F. Let T and U be linear transformations from V into W. The function $(T+U)$ defined by

$$(T+U)(\alpha) = T\alpha + U\alpha$$

is a linear transformation from V into W. If c is any element of F, the function (cT) defined by

$$(cT)(\alpha) = c(T\alpha)$$

is a linear transfrormation from V into W. The set of all linear transformations from V into W, together with the addition and scalar multiplication defined above, is a vector space over the field F.

Remark :

If one defined sum and scalar multiple as we did above, then **the set of all functions** from V into W becomes a vector space over the field F.

This has nothing to do with the fact that V is a vector space, only that V **is a non-empty set**.

When V is a vector space we can define a linear transformation from V into W, and Theorem (17) says that the linear transformations are a subspace of the space of all functions from V into W.

We denote the space of linear transformations from V into W by $L(V, W)$.

Reminder : $L(V, W)$ is defined only when V and W are **vector spaces over the same field**.

The Algebra of Linear Transformations

Theorem 18.

Let V be an n-dimensional vector space over the field F, and let W be an m-dimensional vector space over F. Then the space $L(V, W)$ is finite-dimensional and has dimension mn.

Theorem 19.

Let V, W and Z be vector spaces over the field F. Let T be a linear transformation from V into W and U a linear transformation from W into Z. Then the composed function UT defined by

$$(UT)(\alpha) = U(T(\alpha))$$

is a linear transformation from V into Z.

We are now concerned with linear transformation of a vector space into itself. Since we would so often have to write 'T is a linear transformation from V into V,' we shall replace this with 'T is a linear operator on V.'

Definition 20.

If V is a vector space over the field F, a **linear operator on** *V is a linear transformation from V into V.*

In the case of Theorem (19) when $V = W = Z$, we see that the composition UT is again a linear operator on V. Thus the space $L(U, V)$ has a "multiplication" defined on it by composition.

In this case the operator TU is also defined, and in general $UT \neq TU$. If T is a linear operator on V, then we can compose T with T, denoted by $T^2 = TT$, and in general $T^n = T \cdots T(n \text{ times})$ $n = 1, 2, 3, \ldots$. We define $T^0 = T$ if $T \neq 0$.

Lemma 21.

Let V be a vector space over the field F ; let U, T_1 and T_2 be linear operators on V ; let c be an element of F.

(i) $IU = UI = U$;

(ii) $U(T_1 + T_2) = UT_1 + UT_2; \quad T_1U + T_2U$;

(iii) $c(UT_1) = (cU)T_1 = U(cT_1)$.

Lemma (21) and Theorem (19) tell us that the vector space $L(V, V)$, together with the composition operation, is what is known as a **linear algebra with identity**.

Example 22.

If A is an $m \times n$ matrix with entries in F, we have the linear transformation T defined by $T(X) = AX$, from $F^{n\times 1}$ into $F^{m\times 1}$. If B is a $p \times m$ matrix, we have the linear transformation U from $F^{m\times 1}$ into $F^{p\times 1}$ defined by $U(Y) = BY$. The composition UT is easily described :

$$\begin{aligned}(UT)(X) &= U(T(X)) \\ &= U(AX) \\ &= B(AX) \\ &= (BA)X.\end{aligned}$$

Thus UT is 'left multiplication by the product matrix BA.'

Example 23.

Let F be a field and V the vector space of all polynomial functions from F into F. Let D be the differentiation operator and let T be the linear operator 'multiplication by x' :

$$(Tf)(x) = xf(x).$$

Then $DT \neq TD$ but $DT - TD = I$, the identity operator.
Even though the 'multiplication' we have on $L(V, V)$ is not commutative, it is nicely related to the vector space operations of $L(V, V)$.

Exercise 24.

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for a vector space V. Consider the linear operators $E^{p,q}$

$$E^{p,q}(\alpha_i) = \delta_{iq}\alpha_p.$$

(i) *Prove that these n^2 linear operators form a basis for the space of linear operators on V.*

(ii) *Let T be a linear operator on V. If $A_j = [T\alpha_j]_B = [A_1, A_2, \ldots, A_n]$, then prove that*

$$T = \sum_p \sum_q A_{pq} E^{p,q}.$$

(iii) *If $U = \sum_r \sum_s B_{rs} E^{r,s}$ is another linear operator on V, then prove that the effect of composing T and U is to multiply the matrices A and B.*

Definition 25.

Let V and W be vector spaces over the field F. Let T be a linear transformation from V into W. The function T from V into W is called **invertible** *if there exists a function U from W into V such that UT is the identity function on V (T has a left-inverse)* **and** *TU is the identity function on W (T has a right-inverse).*

If T is invertible, the function U is unique and is denoted by T^{-1}.

Exercise 26.

T is invertible iff

(1) *T is one-to-one ($T\alpha = T\beta$ implies $\alpha = \beta$)*

(2) *T is onto (the range of T is all of W).*

Theorem 27.

Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. If T is invertible, then the inverse function T^{-1} is a linear transformation from W onto V.

Exercise 28.

Let T and U be invertible linear transformations from V onto W and from W onto Z respectively. Then prove that UT is invertible and

$$(UT)^{-1} = T^{-1}U^{-1}.$$

[Hint : It is enough to verify that $T^{-1}U^{-1}$ is both a left and a right inverse for UT.]

Definition 29.

A linear transformation T is **non-singular** *if $T\gamma = 0$ implies $\gamma = 0$, that is, if the null space of T is $\{0\}$.*

Evidently, T is one-to-one iff T is non-singular.

The extension of this remark is that non-singular linear transformations are those which preserve linear transformations.

Theorem 30.

Let T be a linear transformation from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

Let F be a subfield of the complex numbers and V be the space of polynomial functions over F. Consider the differentiation operator D and the "multiplication by x" operator T :

$$(Tf)(x) = xf(x).$$

1. V is not finite-dimensional.
2. Since D sends all constants into 0, D is singular.
3. $R(D) = V$; it is possible to define a **right inverse** of D.
4. The indefinite operator E defined by

$$E(c_0 + c_1 x + \cdots + c_n x^n) = c_0 x + \frac{1}{2}x^2 + \cdots + \frac{1}{n+1} c_n x^{n+1}$$

   is a linear operator on V and $DE = I$. The indefinite integral operator is a right inverse of differentiation operator.

5. On the other hand, $ED \neq I$ because ED sends the constants into 0.
6. $N(T) = V$; it is possible to define a left inverse of T.
7. If U is the operation "remove the constant term" and divide by x

$$U(c_0 + c_1 x + \cdots + c_n x^n) = c_1 + c_2 x + \cdots + c_n x^{n11}$$

   then U is a linear operator on V and $UT = I$.
8. But $TU \neq I$, since every function in the range of TU is in the range of T, which is the space of polynomial functions f such that $f(0) = 0$.
9. This example illustrates that a linear transformation **may be non-singular (one-to-one) without being onto, or may be onto without being non-singular (one-to-one)**. But this cannot happen when dimensions of V and W are same.

Theorem 31.

Let V and W be finite-dimensional vector spaces over the field F such that dim $V =$ dim W. If T is a linear transformation from V into W, the following are equivalent :

(i) *T is invertible.*

(ii) *T is non-singular.*

(iii) *T is onto, that is, the rnage of T is W.*

The set of **invertible operators on a space** V with the operation of composition, is a group.

Exercises 32.

1. *Let T and U be the linear operators on $\mathbb{R}^2$ defined by* $T(x_1, x_2) = (x_2, x_2) \quad U(x_1, x_2) = (x_1, 0)$.
   (i) *How would you describe T and U geometrically ?*
   (ii) *Give rules like the ones defining T and U for each of the transformations* $(U + T), UT, TU, T^2, U^2$.
2. *Let T be the (unique) linear operator on $\mathbb{C}^2$ for which*

$$Te_1 = (1, 0, i), \quad Te_2 = (0, 1, 1), \quad Te_3 = (i, 1, 0).$$

   Is T invertible?
3. *Let T be the linear operator on $\mathbb{R}^3$ defined by*

$$T(x_1, x_2, x_3) = (3x_1, x_1 - 2x_2, 2x_1 + x_1 + x_3).$$

   Is T invertible? If so, find a rule T^{-1} like the one which defines T. Prove that $(T^2 - I)(T - 3I) = 0$.

Exercises 33.

1. *Let $\mathbb{C}^{2\times 2}$ be the complex vector space of 2×2 matrices with complex entries. Let $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$ and let T be the linear operator on $\mathbb{C}^{2\times 2}$ defined by $T(A) = BA$. What is the rank of T ? Can you describe T^2?*
2. *Let T be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$. Prove that the transformation UT is not invertible. Generalize the theorem.*
3. *Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.*
4. *Let V be a vector space over the field F and T a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator T on $\mathbb{R}^2$ such that $T^2 = 0$ but $T \neq 0$.*

Exercises 34.

1. *Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite-dimensional.*
   [Hint : Let $T = D$, the differentiation operator on the space of polynomial functions.]
2. *Let A be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n \times 1}$ into $F^{m \times 1}$ defined by $T(X) = AX$. Show that if $m < n$ it may happen that T is onto without being non-singular. Similarly, show that if $m > n$ we may have T non-singular but not onto.*

Exercises 35.

1. *Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that rank $(T^2) =$ rank (T). Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.*
2. *Let p, m, and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and W the space of $p \times n$ matrices over F. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.*

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