Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal
Numerical Analysis - MA 704
Problem Sheet 1

Dr. P. Sam Johnson (nitksam@gmail.com) http://sam.nitk.ac.in/

1. Show that the function $f(x) = x\sin(1/x)$, with $f(0) = 0$, is continuous at 0 but not differentiable at 0.
2. Show that $f(x) = x^2\sin(1/x)$, with $f(0) = 0$, is once differentiable at 0 but not twice.
   What about the function $x^n|x|$, for some positive integer n?
3. Let $f(x) = x^{-3}(x - \sin x)$ for $x \neq 0$. How should $f(0)$ be defined in order that f be continuous? Will it also be differentiable?
   (a) Find $\max_{a \leq x \leq b} |f(x)|$ for the functions and intervals: $f(x) = 1 + e^{-\cos(x-1)}$, $[1, 2]$ and $f(x) = (4x-3)/(x^2-2x)$, $[0.5, 1]$.
   (b) Check whether $f'(x)$ is 0 at least once in the interval $[1, 2]$ where $f(x) = x \in \pi x - (x-2)\ln x$.
4. For the function $f(x) = 3 - 2x + x^2$ and the interval $[a, b] = [1, 3]$, find the number ξ that occurs in the Mean Value Theorem.
5. Find the Taylor series for $f(x) = \cosh x$ about the point $c = 0$.
6. If the series for $\ln x$ is truncated after the term involving $(x-1)^{1000}$ and is then used to compute $\ln 2$, what bound on the error can be given?
7. Find the Taylor series for $f(x) = e^x$ about the point $c = 3$. Then simplify the series and show how it could have been obtained directly from the series for f about $c = 0$.
8. State and prove generalized Rolle's theorem.
9. For small values of x, the approximation $\sin x \approx x$ is often used. Estimate the error in using this formula with the aid of Taylor's Theorem. For what range of values of x will this approximation give resutls correct to six decimal places?
10. Use Taylor's Theorem with $n = 2$ to prove the inequality $1 + x < e^x$ is valid for all real numbers except $x = 0$.
11. What is the third term in the Taylor expansioin of $x^2 + x - 2$ about the point 3?
12. Determine the first two terms of the Taylor series for x^x about 1 and the remainder term E_1.
13. First develop the function $\sqrt{x}$ in a series of powers of $(x-1)$ and then use it to approximate $\sqrt{0.99999\,99995}$ to ten decimal places.
14. Determine a function that can be termed the **linearization** of $x^3 - 2x$ at 2.
15. How many terms are required in the series
$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$
   to give e with an error of at most $6/10$ unit in the 20th decimal place?
16. Assume $f(x)$ is continuous on $a \leq x \leq b$, and consider the average
$$S = \frac{1}{n}\sum_{j=1}^{n} f(x_j)$$
   with all points x_j in the interval $[a, b]$. Show that for some ξ in $[a, b]$.

17. Derived the inequality :

$$|x-z| \le |\tan x - \tan z| \quad -\frac{1}{\pi} < x, z < \frac{\pi}{2}.$$

18. Assuming $g \in C[a,b]$, show

$$\int_0^h x^2(h-x)^2 g(x)dx = \frac{h^5}{30} g(\xi)$$

for some ξ in $[a,b]$.

19. Construct a Taylor series for the following functions, and bound the error when truncating after n terms.

    (a) $\cos x + \sin x$

    (b) $\frac{1}{x}\int_0^x \frac{\tan^{-1} t}{t} dt.$

20. (a) Prove that for $|x| < 1$,

$$\tan^{-1} x = \sum_{j=1}^{n} (-1)^{j+1} \frac{x^{2j-1}}{2j-1} + (-1)^{j+1} \int_0^x \frac{u^{2n+2}}{1+u^2} du.$$

    (b) Using (20a), show that

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$$

    and we can obtain π by multiplying by 4. Why it this not a practical way to computer π?

    (c) Using a Taylor polynomial approximation, give a practical way to evaluate π.

21. Using Taylor's theorem for functions of two variables, find linear and quadratic approximations to the following functions $f(x,y)$ for small values of x and y. Give the tangent plane function $z = p(x,y)$ whose graph is tangent to that of $z = f(x,y)$ at $(0,0,f(0,0))$.

    (a) $\sqrt{1+2x-y}$

    (b) $\cos(x+\sqrt{\pi^2+y})$

22. Suppose $f \in C[a,b]$, that x_1 and x_2 are in $[a,b]$, and that c_1 and c_2 are positive constants. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

23. A rectangular parallelepiped has sides of length 3cm, 4cm, and 5cm, measured to the nearest centimeter. What are the best upper and lower bounds for the volume of this parallelepiped? What are the best upper and lower bounds for the surface area?

24. The sequnce (F_n) described by $F_0 = 1, F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$, if $n \ge 0$, is called the *Fibonacci squence*. Its terms occur naturally in many botanicak species, particularly those with petals or scales arranged in the form of a logarithmic spiral. Consider the sequence (x_n), where $x_n = F_{n+1}/F_n$. Assuming that $\lim_{n\to\infty} x_n = x$ exists, show that $x = (1+\sqrt{5})/2$. This number is called the *golden ratio*.

25. Use a Taylor polynomial about $\pi/4$ to approximate $\cos 42^\circ$ to an accuracy of 10^{-16}.

26. A function $f : [a,b] \to \mathbb{R}$ is said to satisfy a *Lipschitz* constant L on $[a,b]$ if, for every $x, y \in [a,b]$, we have $|f(x) - f(y)| \le L\,|x-y|$.

    (a) Show that if f satisfies a Lipschitz condition with Lipschitz constant L on an interval $[a,b]$, then $f \in C[a,b]$.

    (b) Show that if f has a derivative that is bounded on $[a,b]$ by L, then f satisfies a Lipschitz condition with Lipschitz constant L on $[a,b]$.

    (c) Give an example of a function that is continuous on a closed interval but does not satisfy a Lipschitz condition on the interval.