Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

P. Sam Johnson nitksam@gmail.com

Computational Linear Algebra - MA 703
Problem Sheet 2

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1. Construct a 3×3 nonzero matrix A such that the vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is a solution of $Ax = 0$.

2. Construct three different linear systems $Ax = b$ whose solution set is $x_1 = -2$, $x_2 = 1$ and $x_3 = 0$.

3. Find all the values of a if the following systems $x + y = 1; \quad 2x + ay = 2$

   (a) has only one solution, (b) infintely many solutions, (c) no solution.

4. Choose h and k such that the system $x + hy = 2p; \quad 4x + 8y = k$.

   (a) has only one solution, (b) infintely many solutions, (c) no solution.

5. Under what condition on y_1, y_2, y_3 do the points $(0, y_1), (1, y_2), (2, y_3)$ lie on a straight line. What is an appropriate generalization of the result?

6. Use Gaussian elimination to find a polynomial which passes through the following points (0,0), (1,4), (-1,0) and (-2,10).

7. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d).

8. Find LU factorization of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$.

9. Find LDU factorization of the matrix $\begin{pmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{pmatrix}$.

10. Find $PA = LU$ factorization of the matrix $\begin{pmatrix} -5 & 3 & 4 \\ 0 & 0 & -9 \\ 15 & 1 & 2 \end{pmatrix}$.

11. Factor the following *tridiagonal* matrices into LU and LDU : $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} a & a & 0 \\ a & a+b & b \\ 0 & a & b+c \end{pmatrix}$.

12. If $A = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$ and $AB = \begin{pmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{pmatrix}$, determine the first and second columns of B.

13. Let $S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Compute the least positive integer k such that S^k is the zero matrix.

14. Find E^2, E^8 and E^{-1} if $E = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$.

15. Let $P_{n\times n}$ be any permutation matrix. Prove that $P^m_{n\times n} = I_{n\times n}$ for some m.

16. Given A, find the LU factorization of A^TA and AA^T and compare the factorizations.

17. Let $x = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ and $A = \begin{pmatrix} 5 & 8 & 7 \\ 0 & 1 & -1 \\ 1 & 3 & 0 \end{pmatrix}$. Is x in the column space of A? why or why not?

18. Invert the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ by the Gauss-Jordan method.

19. Let $A = \begin{pmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{pmatrix}$. Find the third column of A^{-1} with out computing the other columns.

20. Let A be an invertible $n \times n$ matrix, and let B be an $n \times n$ matrix. If $[A \quad B]$ is row equivalent of $[I \quad X]$, what is the relation between X, A and B?

21. Let $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -5 \\ 3 & k \end{pmatrix}$. What value(s) of k, if any, will make $AB = BA$?

22. Could a set of three vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$? What about n vectors in $\mathbb{R}^m$ when in n is less than m?

23. Let A be a 3×4 matrix, let y_1 and y_2 be vecotrs in $\mathbb{R}^3$, and let $w = y_1 + y_2$. Suppose $y_1 = Ax_1$ and $y_2 = Ax_2$ for some vectors x_1 and x_2 in $\mathbb{R}^4$. What fact allows you to conclude that the system $Ax = w$ is consistent?

24. Give all the 2×2 matrices A such that $A^2 = I$.

25. Let A, B and C be matrices with suitable sizes. Prove that $(AB)C = A(BC)$, the matrix multiplication is associative.

26. Choose the only B (3 by 3 matrix) so that for every matrix A,
    (a) BA has rows 1 and 3 of A reversed and row 2 unchanged.
    (b) $BA = 4A$.
    (c) All rows of BA are the same as row 1 of A.
    (d) $BA = 4B$.

27. What rows or columns or matrices do you multiply to find
    (a) the third column of AB?
    (b) the first row of AB?
    (c) the entry in row 3, column 4 of AB?
    (d) the entry in row 1, column 1 of CDE?

28. If you multiply a *northwest matrix* A and a *southeast matrix* B, what type of matrices are AB and BA? "Northwest" and "southeast" mean zeros below and above the antidiagonal going from $(1, n)$ to $(n, 1)$.

29. *Elimination for a 2 by 2 block matrix:* When $A^{-1}A = I$, multiply the first block row by CA^{-1} and substract from the second row, to find the *"Schur complement" S*:

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & S \end{pmatrix}.$$

30. Invent a 3 by 3 **magic matrix** M with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be $8, 3, 4$. What is M times $(1, 1, 1)$? What is the row vector $[1\ 1\ 1]$ times M?

31. Find all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that satisfy $A \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A$.

32. Let $V = \mathbb{R}^n$ and A be a $n \times n$ matrix. If $Ax = 0$ has a unique solution then $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.