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Computational Linear Algebra - MA 703
Problem Sheet 1

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1. In each of the following, find precisely which axioms in the definition of a vector space are violated. Take $V = \mathbb{R}^2$ and $F = \mathbb{R}$ throughout
   (a) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0), \quad \alpha(x_1, x_2) = (\alpha x_1, 0)$
   (b) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha x_1, 0)$
   (c) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha x_1, 2\alpha x_2)$
   (d) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha + x_1, \alpha + x_2)$.

2. True or False : The set of all positive real numbers forms a vector space over $\mathbb{R}$ if the sum of x and y is defined to be the usual product xy and α times x is defined to be x^α.

3. Let V be a vector space. On $V \times V$, define $+$, and . as follows:
$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + y_1, x_2 + y_2)\\ \alpha(x, y) &= (\alpha x, \alpha y), \alpha \in \mathbb{R}, x, y \in V.\end{aligned}$$
   Is $V \times V$ a vector space? If not, write down the conditions (axioms) which are violated.

4. Let $X := \{*\}$ be a singleton set and let V be a vector space. Let $W = \{*\} \times V$. Can we turn W into a vector space as follows?
$$\begin{aligned}(*, x_1) + (*, x_2) &= (*, x_1 + x_2), x_1, x_2 \in V\\ \alpha(*, x) &= (*, \alpha x), \alpha \in \mathbb{R}, x \in V.\end{aligned}$$

5. Prove or disprove: $\mathrm{Sp}(A) \cap \mathrm{Sp}(B) \neq \{0\} \implies A \cap B \neq \emptyset$.

6. True or False : If $A \subseteq B$ and $\mathrm{Sp}(A) \supseteq B$, then $\mathrm{Sp}(A) = \mathrm{Sp}(B)$.

7. If x and y are linearly independent show that $x + \alpha y$ and $x + \beta y$ are linearly independent whenever $\alpha \neq \beta$.

8. Let $\mathtt{Sp}(A) = S$. Then show that no proper subset of A generates S iff A is linearly independent.

9. For what values of α are the vectors $(0, 1, \alpha), (\alpha, 1, 0)$ and $(1, \alpha, 1)$ in $\mathbb{R}^3$ linearly independent.

10. Given W_1, W_2 vector subspaces of V, does there exist any smallest vector subspace W_3 containing W_1 and W_2?

11. Let W be a vector subspace of V. What is $w + W$ if $w \in W$? What is $W + W$? Is is true that $w + W = W$ if and only if $w \in W$?

12. Say true or false: If x and y are linearly independent vectors in V, then so are $x + y$ and $x - y$.

13. Prove or disprove: If A, B and C are pair-wise disjoint subsets of V such that $A \cup B$ and $A \cup C$ are bases of V, then $\mathtt{Sp}(B) = \mathtt{Sp}(C)$.

14. Let $x_1, x_2, \ldots, x_n$ be fixed distinct real numbers.

 (a) Show that $\ell_1(t), \ell_2(t), \ldots, \ell_n(t)$ form a basis of $\mathcal{P}_n$, where $\ell_i(t) = \Pi_{i \neq j}(t - x_j)$. This basis leads to what is known as *Lagrange's interpolation formula*. If $f(t) \in \mathcal{P}_n$ is written as $\sum_{i=1}^{n} \alpha_i \ell_i(t)$, show that $\alpha_i = f(x_i)/\ell_i(x_i)$.

 (b) Show that $\psi_1(t), \psi_2(t), \ldots, \psi_n(t)$ form a basis of $\mathcal{P}_n$, where $\psi_i(t) = 1$ and $\psi_i(t) = \Pi_{j=1}^{i-1}(t - x_j)$ for $i = 2, \ldots, n$. This basis leads to what is known as *Newton's divided difference formula*.

15. Let V be a vector space. Prove that arbitrary intersection of subspaces of V is again a subspace of V. Is a union of two subspaces again a subspace?

16. Extend $A = \{(1, 1, \ldots, 1)\}$ to a basis of $\mathbb{R}^n$.

17. Let S and T be subspaces of a vector space V with $d(S) = 2, d(T) = 3$ and $d(V) = 5$. Find the minimum and maximum possible values of $d(S + T)$ and show that every (integer) value between these can be attained.

18. Show that the distributive law

$$S \cup (T + W) = (S \cup T) + (S \cup W)$$

is false for subspaces. However prove that it holds whenever $S \supseteq T$ or $S \supseteq W$. This latter result is known as the *modular law*.

19. The sum of two subspaces S and T is said to be *direct* (or S and T *independent*) if any vector in $S + T$ can be expressed in a unique way as $x + y$ with $x \in S$ and $y \in T$. Prove that the following statements are equivalent.

 (a) $S + T$ is direct.

 (b) $S \cup T = \{0\}$.

 (c) If $x \in S - \{0\}$ and $y \in T - \{0\}$, then x, y are linearly independent.

 (d) $0 = x + y, x \in S, y \in T \Rightarrow x = 0$ and $y = 0$.

 (e) $d(S + T) = d(S) + d(T)$.

20. Say true or false: A complement of a subspace is unique.

21. True or False : If $\{x_1, x_2, \ldots, x_k\}$ is a basis of a subspace S, then

 (a) $\{\alpha x_1, x_2, \ldots, x_k\}$ is a basis of S iff $\alpha \neq 0$.

 (b) $\{x_1 + \beta x_2, x_2, \ldots, x_k\}$ is a basis of S for any scalar β.

 (c) $\{x_1 + \beta x_2, \alpha x_1 + x_2, x_3, \ldots, x_k\}$ is a basis of S iff $\alpha\beta \neq 1$.

22. Consider the subspaces

 (a) $S_1 = \{(\alpha, \beta, \alpha, \beta, -2\alpha - 2\beta) : \alpha, \beta \in \mathbb{R}\}$

 (b) $S_2 = \{(\alpha, \alpha, \beta, \beta, -2\alpha - 2\beta) : \alpha, \beta \in \mathbb{R}\}$

 (c) $S_3 = \{(\alpha, \beta, \beta, 2\beta - \alpha, -4\beta) : \alpha, \beta \in \mathbb{R}\}$

 (d) $S_4 = \{(0, \alpha, 0, \beta, -\alpha - \beta) : \alpha, \beta \in \mathbb{R}\}$ of $\mathbb{R}^5$.

Find an ordered basis of $S_1 + \cdots + S_4$ such that the first r_i vectors form a basis of $S_1 + \cdots + S_i$ (for some r_i) for each i.