Numerical Integration

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Overview

We derive and analyse numerical methods for evaluating definite integrals. The integrals are mainly of the form

$$\mathtt{I}(f) = \int_a^b f(x)dx.$$

Most such integrals cannot be evaluated explicitly. It is often faster to integrate them numerically rather than evaluating them exactly using a complicated antiderivative of $f(x)$.

The approximation of $\mathtt{I}(f)$ is usually referred to as **numerical integration** or **quadrature.**

Overview

There are many numerical methods for evaluating

$$\mathtt{I}(f) = \int_a^b f(x)dx,$$

but most can be made to fit within the following simple framework.

For the integrand $f(x)$, find an approximating family $\{f_n(x) : n \geq 1\}$ and define

$$\mathtt{I}_n(f) = \int_a^b f_n(x)\ dx = \mathtt{I}(f_n).$$

We usually require the approximations $f_n(x)$ to satisfy

$$\|f - f_n\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Overview

For the error,

$$\begin{aligned} E_n(f) &= \mathtt{I}(f) - \mathtt{I}_n(f) \\ &= \int_a^b [f(x) - f_n(x)]\, dx. \end{aligned}$$

Hence

$$\begin{aligned} |E_n(f)| &\leq \int_a^b |f(x) - f_n(x)|\, dx \\ &\leq (b-a)\ \|f - f_n\|_\infty. \end{aligned}$$

Most numerical integration methods can be viewed within this framework.

Overview

Most numerical integrals $\mathtt{I}_n(f)$ will have the following form when they are evaluated :

$$\mathtt{I}_n(f) = \sum_{j=1}^{n} w_{j,n} f(x_{j,n}) \quad \text{for} \quad n \geq 1.$$

The coefficients $w_{j,n}$ are called the **integration weights** or **quadrature weights**; and the points $x_{j,n}$ are the **integration nodes**, usually chosen in $[a, b]$.

Standard methods have nodes and weights that have simple formulas or else they are tabulated in tables that are readily available.

Overview

Thus there is usually no need to explicitly construct the functions $f_n(x)$ of

$$\mathtt{I}(f_n) = \int_a^b f_n(x)\ dx$$

although their role in defining $\mathtt{I}_n(f)$ may be useful to keep in mind.

Most numerical integration formulas are based on defining $f_n(x)$ in

$$\mathtt{I}(f_n) = \int_a^b f_n(x)\ dx$$

by using polynomial or piecewise polynomial interpolation.

Formulas using such interpolation **with evenly spaced node points** are derived and discussed.

Quadrature Formula

The process of evaluation a definite integral from a set of tabulated values of the integrand $f(x)$ is called **numerical integration**. This process when applied to a function of a single variable is known as **quadrature**.

The problem of numerical integration, like that of numerical differentiation is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits.

In this way, we can derive quadrature formulae for approximate integration of a function defined by a set of numerical values only.

Newton-Cotes Quadrature Formula

Let

$$\text{I} = \int_a^b f(x)dx$$

where $f(x)$ takes the values $y_0, y_1, \ldots, y_n$ for $x = x_0, x_1, \ldots, x_n$.

Let us divide the interval $[a, b]$ into n-subintervals of width h so that $x_0 = a, \quad x_1 = x_0 + h, \quad \ldots, \quad x_n = x_0 + nh = b$. Then

$$\begin{aligned}
\text{I} &= \int_{x_0}^{x_0+nh} f(x)dx \\
&= h\int_0^h f(x_0 + ph)dp \quad \text{putting } x = x_0 + ph \\
&= h\int_0^h \left[y_0 + p \bigtriangleup y_0 + \frac{p(p-1)}{2!} \bigtriangleup^2 y_0 + \ldots\right] dp
\end{aligned}$$

(by Newton's forward interpolation formula)

Newton-Cotes Quadrature Formula

Integration term by term, we obtain,

$$\begin{aligned}\int_{x_0}^{x_0+nh} f(x)dx \quad = \quad & nh\left[y_0 + \frac{n}{2} \triangle y_0 + \frac{n(2n-3)}{12} \triangle^2 y_0 + \frac{n(n-2)^2}{24} \triangle^3 y_0 \right.\\ & + \left(\frac{n^4}{5} - \frac{3n^2}{2} + \frac{11n^2}{3} - 3n\right)\frac{\triangle^4 y_0}{4!} \\ & + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\triangle^5 y_0}{5!} \\ & \left. + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 15n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\triangle^6 y_0}{6!} + \cdots\right]\end{aligned} \tag{1}$$

This is known as **Newton-Cotes quadrature formula.** From this general formula, we deduce the following important quadrature rule by taking $n = 1, 2, 3, \ldots$.

Trapezoidal Rule

Putting $n = 1$ in (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line.

That is, **approximate** y **by a polynomial of first order**, so that differences of order higher than first becomes zero, we get

$$\int_{x_0}^{x_0+h} f(x)dx = h\left(y_0 + \frac{1}{2} \triangle y_0\right) = \frac{h}{2}(y_0 + y_1).$$

Similarly

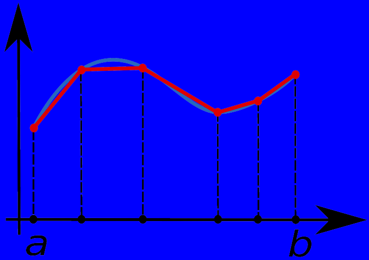
$$\begin{aligned}
\int_{x_0+h}^{x_0+2h} f(x)dx &= h\left(y_1 + \frac{1}{2} \triangle y_1\right) = \frac{h}{2}(y_1 + y_2) \\
&\vdots \\
\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx &= \frac{h}{2}(y_{n-1} + y_n).
\end{aligned}$$

Trapezoidal Rule

Adding these n integrals, we obtain

$$\int_{a=x_0}^{b=x_0+nh} f(x)dx = \frac{h}{2}\Big[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\Big].$$

This is known as the **Trapezoidal rule.**



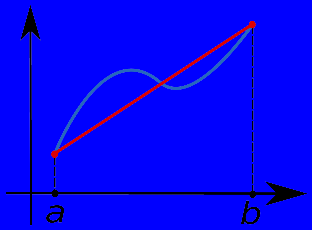
The area of each strip (trapezium is found separately. Hence the area under the curve $y = f(x)$, between the lines $x = x_0, x = x_n$ is approximately equal to the sum of the area of the n trapeziums.

Error in Trapezoidal Rule

The simple trapezoidal rule is based on approximating $f(x)$ by the straight line joining $(a, f(a))$ and $(b, f(b))$.

By integrating the formula for this straight line, we obtain the approximation

$$\mathrm{I}_1(f) = \left[\frac{f(a) + f(b)}{2}\right](b - a).$$



Of course, this is the area of the trapezoid shown in the graph.

Error in Trapezoidal Rule

Note that the equation of the line joining $(a, f(a))$ and $(b, f(b))$ is

$$y = \frac{(b-x)f(a) + (x-a)f(b)}{b-a}.$$

We further assume that $f(x)$ is twice continuously differentiable on $[a, b]$. The error for the trapezoidal rule is

$$\begin{aligned} E_1(f) &= \int_a^b f(x)\, dx - \left\{ \frac{f(a)+f(b)}{2}(b-a) \right\} \\ &= \int_a^b (x-a)(x-b)\ f[a, b, x]\, dx. \end{aligned}$$

Here $f[a, b, x]$ is the 3-rd order Newton divided difference of f about the nodes a, b and x.

Error in Trapezoidal Rule

Using the integral mean value theorem

$$\begin{aligned} E_1(f) &= f[a,b,\xi] \int_a^b (x-a)(x-b)\,dx \quad \text{for some } \xi \in [a,b] \\ &= \left[\frac{1}{2} f''(\eta)\right] \left[\frac{-1}{6}(b-a)^3\right] \quad \text{for some } \eta \in [a,b]. \end{aligned}$$

Thus the error is

$$E_1(f) = \frac{-(b-a)^3}{12} f''(\eta) \quad \text{for some } \eta \in [a,b].$$

If the value $b-a$ is not sufficiently small, the trapezoidal rule is not of much use.

For such case, we break the integral into a sum of integrals over smal subintervals, and then apply the trapezoidal rule to each of these integrals.

Error in Composite Trapezoidal Rule

Let $n \geq 1$, $h = \frac{b-a}{n}$, and $x_j = a + jh$ for $j = 0, 1, 2, \ldots, n$. Then

$$\begin{aligned} \mathtt{I}(f) &= \int_a^b f(x)\, dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\, dx \\ &= \sum_{j=1}^{n} \left\{ \left(\frac{h}{2}\right) [f(x_{j-1}) + f(x_j)] - \frac{h^3}{12} f''(\eta_j) \right\} \end{aligned}$$

with $x_{j-1} \leq \eta_j \leq x_j$.

The first terms in the sum can be combined to give the **composite trapezoidal rule**,

$$\mathtt{I}_n(f) = \frac{h}{2}[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n] \qquad n \geq 1$$

with $f(x_j) \equiv f_j$.

Error in Composite Trapezoidal Rule

The error in $\mathtt{I}_n(f)$ is given by

$$E_n(f) = \mathtt{I}(f) - \mathtt{I}_n(f) = \sum_{j=1}^{n} \frac{-h^3}{12} f''(\eta_j) = \frac{-h^3}{12}\left[\frac{1}{n}\sum_{j=1}^{n} f''(\eta_j)\right].$$

For the term in brackets,

$$\min_{a \le x \le b} f''(x) \le \frac{1}{n}\sum_{j=1}^{n} f''(\eta_j) \le \max_{a \le x \le b} f''(x).$$

Since $f(x)$ is continuous for $a \le x \le b$, it must attain all values between its minimum and maximum at some point of $[a, b]$. Hence

$$f(\eta) = M \qquad \text{for some} \quad \eta \in [a, b].$$

Error in Composite Trapezoidal Rule

Thus we can write

$$\begin{aligned} E_n(f) &= \frac{-(hn)h^2}{12} f''(\eta) \\ &= \frac{-(b-a)h^2}{12} f''(\eta) \end{aligned}$$

for some $\eta \in [a, b]$.

There is no reason why the subintervals $[x_{j-1}, x_j]$ must all have equal length, but it is customary to first introduce the general principles involved in this way.

Although this is also the customary way in which the method is applied, there are situations in which it is desirable to vary the spacing of the nodes.

Simpson's 1/3-rule

Putting $n = 2$ in the Newton-Cotes general quadrature formula and taking the curve through $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) as a parabola, we get

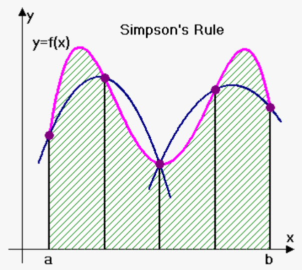
$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \triangle\, y_0 + \frac{1}{6}\,\triangle^2\, y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

As the function $y = f(x)$ in approximated on $[x_0, x_1]$ by a polynomial of degree two and it is passing through the points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2), the interval $[x_0, x_0 + nh]$ should be divided into even number of subintervals, that is , n has to be even.

Simpson's 1/3-rule

Thus,

$$\int_{a=x_0}^{b=x_0+nh} f(x)dx = \frac{h}{3}\Big[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) \\ +2(y_2 + y_4 + \cdots + y_{n-2})\Big].$$



This is known as the **Simpson's one-third rule** or **Simpson's rule** and is most commonly used.

Error in Simpson's 1/3-rule

Let us consider the simple Simpson rule

$$\mathtt{I}_2(f) = \frac{h}{3}\left[(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \qquad \text{where} \quad h = \frac{b-a}{2}.$$

For the error,

$$E_2(f) = \mathtt{I}(f) - \mathtt{I}_2(f) = \int_a^n (x-a)(x-c)(x-b)f[a,b,c,x]\, dx.$$

If we further assume that f is four times continuously differentiable on $[a, b]$, we can apply the integral mean value theorem. Thus

$$E_2(f) = \frac{-h^5}{90} f^{(4)}(\eta) \qquad \text{where} \quad \eta \in [a, b].$$

From this, we see that $E_2(f) = 0$ if $f(x)$ is a polynomial of degree at most three, even though quadratic interpolation is exact only if $f(x)$ is a polynomial of degree at most two. This results in Simpson's rule being much more accurate than the trapezoidal rule.

Error in Composite Simpson's 1/3-rule

For the error, as with the trapezoidal rule

$$E_n(f) = \mathtt{I}(f) - \mathtt{I}_n(f) = \frac{-h^5(n/2)}{90}\frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\eta_j).$$

Since f is 4-times differentiable, we get

$$E_n(f) = \frac{h^4(b-a)}{180} f^{(4)}(\eta) \qquad \text{where} \quad \eta \in [a, b].$$

Simpson's 3/8-rule

Putting $n = 3$ in the general Newton-Cotes quadrature formula and taking the curve through $(x_i, y_i), \quad i = 0, 1, 2, 3$ as a polynomial of third order, we get

$$\int_{x_0}^{x_0+3h} f(x)dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3).$$

When n is a multiple of 3, we obtain

$$\begin{aligned}\int_{x_0}^{x_0+nh} f(x)dx &= \frac{3h}{8}\Big[(y_0 + y_1) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) \\ &\qquad + 2(y_3 + y_6 + \cdots + y_{n-3})\Big],\end{aligned}$$

which is known as **Simpson's 3/8-rule.**

Exercises

1. *Evaluate*
$$\int_0^6 \frac{dx}{1+x^2}$$
by using
   (a) *Trapezoidal rule*
   (b) *Simpson's 1/3-rule*
   (c) *Simpson's 3/8-rule Weddle's rule*

   and compare the results with its actual value.

2. *Evaluate*
$$\int_0^1 \frac{x^2}{1+x^2}\,dx$$
by using Simpson's 1/3- rule. Compare the error with the exact value.

Exercises

3. *Use the Trapezoidal rule to estimate the integral*

$$\int_0^2 e^{x^2}\,dx$$

taking 10 sub-intervals.

4. *Use Simpson's* $1/3$*-rule to find*

$$\int_0^{0.6} e^{-x^2}\,dx$$

by taking seven ordinates. Compare the approximate with the exact value.

5. *Using Simpson's* $3/8$*-th rule, compute the value of*

$$\int_{0.2}^{1.4} (\sin x - \log x + e^x)\,dx.$$

Applications

If the ordinates, $x_0, x_1, \ldots, x_n$ represent equispace cross-sectional areas, then Simpson's rule gives the volume of a solid. As such Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam.

Similarly if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance traveled.

Exercise

6. *The velocity $v(km/min)$ of a moped which starts from rest, in given at fixed intervals of time t (min) as follows*

t :	2	4	6	8	10	12	14	16	18	20
v :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Exercises

7. *The velocity v of a particle at distance s from a point on its linear path is given by the following table:*

$s(m)$:	0	2.5	5.0	7.5	10	12.5	15	17.5	20
$v(m/sec)$:	16	19	21	22	20	17	13	17	9

Estimate the time taken by the particle to traverse the distance of 20 meters, using Boole's value.

8. *A solid of revolution is formed by rotating about the x- axis, the area between the x- axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following co-ordinates.*

x :	0	0.25	0.5	0.75	1
y :	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule.

9. *A river is 80 ft. wide. The depth d in feet at a distance x ft. from one bank is given by the following table. Find approximately the area of the cross-section.*

x :	0	10	20	30	40	50	60	70	80
y :	0	4	7	9	12	15	14	8	3

Exercises

10. *A body is in the form of a solid of revolution. The diameter D is cm. of its sections at distances x cm. from on end are given below. Estimate the volume of the solid.*

x :	0	2.5	5	7.5	10	12.5	15
D :	5	5.5	6	6.75	6.25	5.5	4

11. *A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's 1/3-rd rule, find the velocity of the rocket at $t = 80$ seconds.*

$t(sec)$:	6	10	20	30	40	50	60	70	80
$f(cm/sec^2)$	30	31.63	33.34	35.47	37.75	40.33	43.23	46.69	50.67

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