Results from Calculus

P. Sam Johnson

August 26, 2014

Taylor's theorem in various forms is fundamental to many numerical procedures and is an excellent starting point for the study of numerical methods since no advanced mathematical concepts are required. The theorem gives a relatively simple method for approximating functions $f(x)$ by polynomials, and thereby gives a method for computing $f(x)$.

Taylor's theorem and the associated Taylor series will be used throughout the course often.

Taylor's Theorem with Integral Remainder

Theorem

Let $f(x)$ have $n+1$ continuous derivatives on $[a, b]$ for some $n \geq 0$, and let $x, x_0 \in [a, b]$. Then $f(x) = P_n(x) + R_n(x)$ where

$$P_n(x) = \sum_{k=0}^{n} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0) \qquad \textit{(n-degree polynomial)}$$

and

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} (x - t)^n f^{(n+1)}(t) dt \qquad \textit{(error term)}$$

for some ξ between x and x_0.

Hence "ξ between x and x_0" means that either $x_0 < \xi < x$ or $x < \xi < x_0$ depending on the particular values of x and x_0 involved.

Here $P_n(X)$ is called the n**th Taylor polynomial** for f about x_0, and R_n is called the **remainder term** (or **truncation error**) associated with $P_n(x)$. Since the number ξ in the truncation error R_n depends on the value of x at which the polynomial $P_n(x)$ is being evaluated, it is a function of the variable x.

Taylor's Theorem simply ensures that such a function exists, and that its value lies between x and x_0. In fact, one of the common problems in numerical methods is to try to determine a realistic bound for the value of $f^{(n+1)}(\xi)$ when x is within some specified interval.

Proof of Taylor's Theorem

Recall the formula for integration by parts

$$\int u\ dv = uv - \int v\ du$$

and apply it to the integral E_n with

$$u = \frac{(x-t)^n}{n!} \quad dv = f^{(n+1)}(t)dt.$$

The result is $R_n = -\frac{1}{n!} f^{(k)}(x_0)(x-x_0)^n + R_{n-1}$. If this process of integrating by parts is repeated, we eventually obtain

$$R_n(x) = -\sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k + R_0.$$

Since $R_0 = \int_{x_0}^{x} f'(t)dt = f(x) - f(x_0)$, we have

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k + R_n.$$

Using Taylor's theorem, we obtain the following standard formulas:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{\xi_x} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \\
&\qquad + (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos(\xi_x) \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \\
&\qquad + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cos(\xi_x)
\end{aligned}$$

For all cases, the unknown point ξ_x is located between 0 and x.

$$(1+x)^{\alpha} = 1 + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \cdots + \binom{\alpha}{n} x^n + \binom{\alpha}{n+1} \frac{x^{n+1}}{(1+\xi_x)^{n+1-\alpha}}$$

with

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad k = 1, 2, 3, \ldots$$

for any real number α.

For all cases, the unknown point ξ_x is located between 0 and x.

- An important special case

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x}, \quad x \neq 1. \qquad (1)$$

Rearranging (1), we obtain the familiar formula for a finite geometric series:

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}, \quad x \neq 1. \qquad (2)$$

- The infinite series obtained by taking the limit of $P_n(x)$ as $n \to \infty$ is called the **Taylor series** for f about x_0. In the case $x_0 = 0$, the Taylor polynomial is often called a **Maclaurin polynomial**, and the Taylor series is often is called a **Maclaurin series**.
- The term truncation error in the Taylor polynomial refers to the error involved in using a truncated, or finite, summation to approximate the sum of an infinite series.
- Infinite series representations for the functions $e^x, \cos x, \sin x, (1+x)^\alpha$ can be obtained by setting $n \to \infty$.
- Infinite series for $e^x, \cos x, \sin x$ converge for all x, and for the functions $(1+x)^\alpha$ and $\frac{1}{1-x}$ converge for $|x| < 1$.
- Formula (2) leads to the well known infinite geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1.$$

Taylor's Theorem with Lagrange Remainder

Theorem

Let $f(x)$ have $n+1$ continuous derivatives on $[a,b]$ for some $n \geq 0$, and let $x, x_0 \in [a,b]$. Then $f(x) = P_n(x) + E_n(x)$ where

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) \qquad \text{(n-degree polynomial)}$$

and

$$E_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \qquad \text{(error term)}$$

for some ξ between x and x_0.

Hence "ξ between x and x_0" means that either $x_0 < \xi < x$ or $x < \xi < x_0$ depending on the particular values of x and x_0 involved.

Other Forms of Taylor's Theorem

Theorem

Let $f(x)$ have $n+1$ continuous derivatives on $[a, b]$ for some $n \geq 0$, and let $x, x+h \in [a, b]$. Then $f(x+h) = P_n(x) + E_n(x)$ where

$$P_n(x) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x) \qquad \text{(n-degree polynomial)}$$

and

$$E_n(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \qquad \text{(error term)}$$

for some ξ between x and $x+h$.

Exercises

1. Determine the Taylor series of $f(x) = \log x$ in the interval $[1, 2]$ about $x = 1$.
2. How many terms in the series need to be used to compute
   (a) $\log 2$ with accuracy of one part in 10^8 ?
   (b) $\log 1.5$ with accuracy of one part in 10^8 ?
3. Find the Taylor series of
   (a) $f(x) = e^{-x^2}$
   (b) $f(x) = \tan^{-1} x$
   (c) $f(x) = \int_0^1 \sin(xt)dt$.

For vector-valued functions of vectors, there also exist Taylor series and Taylor formulas. Thus, if f is a mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$, formulas are available for expressing $f(x+h)$ in terms of $f(x), f'(x), f''(x)$, and so on. Of course, the main difficulty lies in defining the appropriate derivatives.

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, the simplest expression of Taylor's formula is derived: Let $f(x, y)$ be a given function of the two independent variables x and y. We will show how the earlier Taylor's theorem can be extended to the expansion of $f(x, y)$ about a given point (x_0, y_0). The results will easily extend to functions of more than two variables. As notation, let $L(x_0, y_0; x_1, y_1)$ denote the set of all points (x, y) on the straight line segment joining (x_0, y_0) and (x_1, y_1).

Taylor's Theorem in Two Dimensions

Theorem

Let (x_0, y_0) and $(x_0 + \xi, y_0 + \eta)$ be given points, and assume $f(x, y)$ is $n + 1$ times continuously differentiable for all (x, y) in some neighborhood of $L(x_0, y_0; x_0 + \xi, y_0 + \eta)$. Then

$$f(x_0 + \xi, y_0 + \eta) = f(x_0, y_0) + \sum_{j=1}^{n} \frac{1}{j!} \left[\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right]^j f(x, y)\Big|_{x=x_0, y=y_0} + \frac{1}{(n+1)!} \left[\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right]^{n+1} f(x, y)\Big|_{x=x_0+\theta\xi, y=y_0+\theta\eta}$$

for some $0 \leq \theta \leq 1$. The point $(x_0 + \theta\xi, y_0 + \theta\eta)$ is unknown point on the line $L(x_0, y_0; x_0 + \xi, y_0 + \eta)$.

Exercises

Using Taylor's theorem for functions of two variables, find linear and quadratic approximations to the following functions $f(x,y)$ for small values of x and y. Give the tangent plane function $z = p(x,y)$ whose graph is tangent to that of $z = f(x,y)$ at $(0,0,f(0,0))$.

1. $f(x,y) = \sqrt{1+2x-y}$
2. $g(x,y) = \frac{1+x}{1+y}$
3. $h(x,y) = x\cos(x-y)$
4. $k(x,y) = \cos(x+\sqrt{\pi^2+y})$