Fixed Point Iteration Method

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The point p is a fixed point of the function g if $g(p) = p$.

We consider a problem of finding "fixed points of a function", called fixed point problem.

Example

The function $f(x) = x^2$ *has fixed points* 0 *and* 1. *Whereas the function* $g(x) = x + 2$ *has no fixed point.*

Root-finding problems and fixed-point problems are equivalent classes in the following sence.

Theorem

f *has a root at* α *iff* $g(x) = x - f(x)$ *has a fixed point at* α.

Several g may exist

There is more than one way to convert a function that has a root at α into a function that has a fixed point at α.

Example

The function $f(x) = x^3 + 4x^2 - 10$ *has a root somewhere in the interval* $[1, 2]$. *Here are several functions that have a fixed point at that root.*

$$
\begin{aligned}
g_1(x) &= x - f(x) = x - x^3 - 4x^2 + 10 &(1)\\
g_2(x) &= \sqrt{\frac{10}{x} - 4x} &(2)\\
g_3(x) &= \frac{1}{2}\sqrt{10 - x^3} &(3)\\
g_4(x) &= \sqrt{\frac{10}{4 - x}} &(4)\\
g_5(x) &= x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} &(5)
\end{aligned}
$$

Sufficient Conditions

Theorem (Existence of a Fixed Point)

If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point.

Theorem (Uniqueness of a Fixed Point)

If g has a fixed point and if $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k \text{ for all } x \in (a,b),$$

then the fixed point in $[a,b]$ is unique.

The condition in the above theorem, is not necessary.

Example

The function $g(x) = 3^{-x}$ on $[0,1]$ has a unique fixed point. But $|g'(x)| \nleq 1$ on $(0,1)$.

If sufficient conditions are satisfied, then how to find the fixed point?

To approximate the fixed point of a function g, we choose an initial approximation x_0 and generate the sequence $(x_n)_{n=0}^{\infty}$ by letting $x_n = g(x_{n-1})$, for each $n \geq 1$.

If the sequence converges to α and g is continuous, then

$$\alpha = \lim_{n\to\infty} x_n = \lim_{n\to\infty} g(x_{n-1}) = g(\lim_{n\to\infty} x_{n-1}) = g(\alpha),$$

and a solution to $x = g(x)$ is obtained.

This technique is called fixed-point iteration, or functional iteration.

Fixed-Point Theorem

Theorem

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition, that g' exists on (a, b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k \text{ for all } x \in (a, b).$$

Then, for any number x_0 in $[a, b]$, the sequence defined by

$$x_n = g(x_{n-1}), n \geq 1,$$

converges to the unique fixed point α in $[a, b]$.

Which g is better?

Using the Mean Value Theorem and the fact that $|g'(x)| \leq k$, we have, for each n,

$$|x_n - \alpha| \leq k|x_{n-1} - \alpha|.$$

Applying the above inequality inductively gives

$$|x_n - \alpha| \leq k^n|x_0 - \alpha|.$$

Since $0 < k < 1$, $(x_n)_{n=1}^{\infty}$ converges to α.

The rate of convergence depends on the factor k^n. The smaller the value of k, the faster the convergence, which may be very slow if k is close to 1.

Finally, we have got some clue (!) for g, which should be rejected.

When to stop the procedure if error bound is given?

If we are satisfied with an approximate solution which is in ε-neighbourhood of the exact value α (ε-distance away from the exact value α), then the following inequalities are helpful.

For all $n \geq 1$,

$$|x_n - \alpha| \leq k^n \max\{x_0 - a, b - x_0\} < \varepsilon$$

and

$$|x_n - \alpha| \leq \frac{k}{1-k}|x_n - x_{n-1}| < \varepsilon.$$

Find the difference between two consecutive approximations, $|x_n - x_{n-1}|$. If

$$|x_n - x_{n-1}| < \frac{1-k}{k}\varepsilon,$$

then we can say that x_n is ε-distance away from the exact value α.

References

- Richard L. Burden and J. Douglas Faires, "*Numerical Analysis – Theory ad Applications*", Cengage Learning, New Delhi, 2005.
- Kendall E. Atkinson, "*An Introduction to Numerical Analysis*", John Wiley & Sons, Delhi, 1989.