Linear Independence, Basis, Dimension & Four Fundamental Subspaces

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Linear Dependent Set

For any vectors $u_1, u_2, \ldots, u_n$ we have that $0u_1 + 0u_2 + \cdots + 0u_n = 0$. This is called the **trivial representation** of 0 as a linear combination of $u_1, u_2, \ldots, u_n$.

This motivates a definition of "**linear dependence**". For a set to be linearly dependent, there must exist a non-trivial representation of 0 as a linear combination of vectors in the set.

Definition

A subset S of a vector space V is called **linearly dependent** *if there exist a finite number of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_n$,* **not all zero**, *such that*

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0.$$

Note that the zero on the right is the **zero vector**, not the number zero.

- Any set containing the zero vector is linearly dependent.
- If $m > n$, a set of m vectors in $\mathbb{R}^n$ is dependent.

A subset S of a vector space V is then said to be **linearly independent** if it is not linearly dependent.

In other words, **a set is linearly independent if the only representations of** 0 **as a linear combination of its vectors are trivial representations**.

More generally, let V be a vector space over $\mathbb{R}$, and let $\{v_i : i \in I\}$ be a family of elements of V. The family is **linearly dependent** over $\mathbb{R}$ if there exists a family $\{a_j : j \in J\}$ of elements of $\mathbb{R}$, not all zero, such that $\sum_{j \in J} a_j v_j = 0$, where the index set J is a nonempty, finite subset of I.

A set $\{v_i : i \in I\}$ of elements of V is **linearly independent** if the corresponding family $\{v_i : i \in I\}$ is not linearly dependent.

Exercise

A family is dependent if a member is in the linear span of the rest of the family.

Spanning a Subspace

Definition

A set of vectors S **spans** *a subspace W if $W = \langle S \rangle$; that is, if every element of W is a linear combination of elements of S.*

In other words, we call the **subspace** W **spanned by a set** S if all possible linear combinations produce the space W.

If S spans a vector space V (we denote $\text{Sp}(S) = V$), then every set containing S is also a spanning set of V.

Definition

A set B of vectors in a vector space V is said to be a **basis** *if B is linearly independent and spans V.*

From the definition of a basis B, every element of V can be written as linear combination of elements of B, **in one and only way**.

Definition

The number of elements of a basis B of a vector space V is called the **dimension** *of V.*

Examples

1. *The coordinate vectors $e_1, e_2, \ldots, e_n$ coming from the identity matrix spans $\mathbb{R}^n$. Hence the dimension of $\mathbb{R}^n$ is n*
2. *The vector space $\mathcal{P}(x)$ of all polynomials in x over $\mathbb{R}$ has the (infinite) subset $1, x, x^2, \ldots$ as a basis, so $\mathcal{P}(x)$ has infinite dimension.*

In a subspace of dimension k, no set of more than k vectors can be independent, and no set of fewer than k vectors can span the space.

- Any linearly independent set in V can be extended to a basis, by adding more vectors if necessary.
- Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

Hence basis is a **maximal linearly independent set**, or a **minimal spanning set**.

Four Fundamental Subspaces

Let A be an $m \times n$ matrix of real entries.

1. The **column space** of A, $\mathcal{C}(A)$ is the space spanned by columns of A. That is,
$$\mathcal{C}(A) := \{Ax : x \in \mathbb{R}^n\}.$$
2. The **null space** of A, $\mathcal{N}(A)$ is the solution set of the matrix equation $Ax = 0$. That is,
$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$
3. The **row space** of A, is the space spanned by rows of A. It is same as the column space of A^T. That is,
$$\mathcal{R}(A^T) := \{y^T A : y \in \mathbb{R}^m\}.$$
4. The **left nullspace** of A is the nullspace of A^T. That is,
$$\mathcal{N}(A^T) := \{y \in \mathbb{R}^m : y^T A = 0\}.$$

Echelon Form

A matrix that has undergone Gaussian elimination is said to be **in row echelon form** or, more properly, "**reduced echelon form**" or "**row-reduced echelon form**". Such a matrix has the following characteristics :

1. All **zero rows** are **at the bottom** of the matrix.
2. The **leading entry of each nonzero row** after the **first occurs** to the **right of the leading entry of the previous row**.
3. The **leading entry** in any nonzero row is 1.
4. All entries in the column **above and below a leading** 1 **are zero**.

The Row Space

- Use Gaussian elimination to transform $[A|b]$ into echelon form $[U|c]$. Transforming to the echelon matrix means that we are taking linear combinations of the rows of a matrix A to come up with the matrix U.
- We could reverse the process and get back to A, by row operations again. Therefore, the row space of A equals the row space of U.
- If 2 matrices are row equivalent, then their row spaces are the same.
- The nonzero rows (rewritten as column vectors) of the matrix U form a **BASIS** for the row space.

Example for a basis of the row space of A

$$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is the echelon matrix of } A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}.$$

A basis for the row space of A is the set of non-zero rows of U (rewritten as column vectors). In our example, a basis for the row space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Notice the number of elements in the basis, which is the number of non-zero rows in U, is just the number of pivots.

The Column Space

- The column space of a matrix consists of **ALL** linear combinations of the columns of the matrix.
- Reduce A to the echelon matrix U, by row operations.
- Find the pivot variables $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ where $r = \text{rank}(A)$ is the total number of pivots. A basis for the column space of A is the set of columns $i_1, i_2, \ldots, i_r$ in the original matrix A.
- That is, the columns of the original matrix corresponding to those columns containing **PIVOTS** form a **BASIS** for the column space of the matrix.
- Notice the number of elements in the basis, which is the number of non-zero rows in U, is just the number of pivots. This means **the column space and the row space of a matrix always have the same dimension**.

Example for a basis of the column space of A

$$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is the echelon matrix of } A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}.$$

Since the pivot variables of A are x_1 and x_4, a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right\}.$$

Caution. We had to choose columns from the original matrix A, because Gaussian elimination changes the column picture at each step.

To see this in our example, just note that the columns of U all have a zero in the third entry: if these spanned the column space of A then every column of A would also have to have zero in the third entry–which is false!

Nullspace

- The system $Ax = 0$ is reduced to $Ux = 0$, where U is an echelon matrix, and this process is reversible.
- The nullspace of A is the same as the nullspace of U.
- Find the free variables $x_{j_1}, x_{j_2}, \ldots, x_{j_{n-r}}$ where $n - r$ is the number of columns of A without a pivot. There is a basis for the nullspace of A, with one vector associated to each free variable. Taking each free variable one at a time, set that free variable as 1 and the other free variables as 0. Then solve $Ux = 0$ for the vector x with this choice and put x in the basis. Repeat for each free variable.
- The $n - r$ "special solutions" to $Ux = 0$ provides a **BASIS** for the nullspace.

Example for a basis of the null space of A

$U = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the echelon matrix of $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{pmatrix}$.

The free variables are x_2 and x_3. Taking $x_2 = 1, x_3 = 0$ and $x_2 = 0, x_3 = 1$ and solve $Ux = 0$ for each choice. A basis for the nullspace of is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Left Nullspace

- Use Gaussian elimination to transform $[A|b]$ into echelon form $[U|c]$.
- A basis for the left nullspace of A has $m - r$ vectors, which is the number of zero rows in U.
- For each zero row, put a vector in the basis whose entries are the coefficients of the vector b in the entry of c corresponding to the zero row.

Example for a basis of the left null space of A

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix} = [U \mid c].$$

There is one zero row of U. As an equation the row represents $0 = -2b_1 - b_2 + b_3$ (note we listed the b_is in order). Thus a basis for the left nullspace is

$$\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

In general, the number of elements in this basis will equal the number of zero rows.

Dimensions of Four Fundamental Subspaces

Let A be $m \times n$ with rank r. Using the bases above, we observe the following:

- dim $R(A) =$ dim $R(A^T) = r$ (the number of pivots).
- dim $N(A) = n - r$ (the number of free columns).
- dim $N(A^T) = m - r$ (the number of zero rows).

Notice that the column space and the row space of a matrix have the same dimension (even though the vectors in each subspace live in a different ambient space, $\mathbb{R}^m$ and $\mathbb{R}^n$.)

The nullspace and row space live in $\mathbb{R}^n$; the left nullspace and column space live in $\mathbb{R}^m$.

Fundamental Theorem of Linear Algebra (Part I)

There is an important relationship between the dimensions of the subspaces in each of these pairs of subspaces, which is shown by the following theorem.

Theorem (Fundamental Theorem of Linear Algebra (Part I))

Let A be $m \times n$ with rank r. Then

- $dim\ R(A^T) + dim\ N(A) = n.$
- $dim\ R(A) + dim\ N(A^T) = m.$

We call the dimension of $N(A)$, **nullity** of A.

The first equation is also called the **rank-nullity law, or, rank-nullity theorem** because rank $A = \dim\ R(A^T)$.

To Temember Their Dimensions !

To remember what are the dimensions of the four fundamental subspaces, it is best just to think about where the bases for each subspace comes from :

- The bases for the column space and row space come from the pivots, so the dimension of each of these subspaces is the rank of the matrix.
- The basis for the nullspace comes from the free columns, so the dimension of the nullspace is the number of free columns.
- The basis for the left nullspace is obtained from the zero rows, so the dimension of the left nullspace is the number of zero rows.

Figure : Nullspace and Row Space in $\mathbb{R}^n$

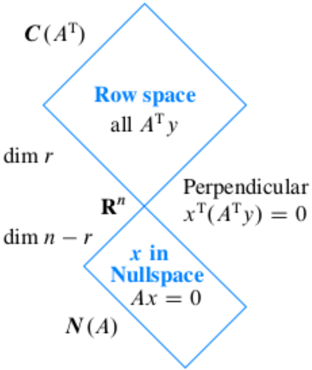
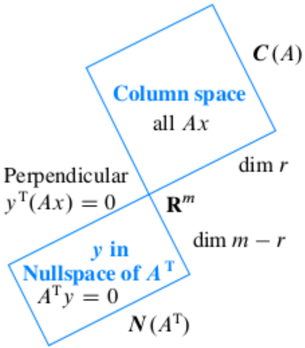


Figure : Left Nullspace and Column Space in $\mathbb{R}^m$



References

- Gilbert Strang, "*Linear Algebra and its Applications*", Cengage Learning, New Delhi, 2006.
- David C. Lay, "*Linear Algebra and its Applications*", Pearson Education, India, 2008.