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Concrete Mathematics - MA 201
Problem Sheet - 1

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1. By the principle of mathematical induction, prove that $T_n = 2^n - 1$ for $n \geq 0$. Here T_n is the recurrence solution of the problem of "Tower of Hanoi".

2. Find the shortest sequence of moves that transfers a tower of n disks from the left peg A to the right peg B, if direct moves between A and B are disallowed. (Each move must be to or from the middle peg. As usual, a larger disk must never appear above a smaller one.)

3. Show that, in the process of transferring a tower under the restrictions of the preceding exercise, we will actually encounter every properly stacked arrangement of n disks on three pegs.

4. Are there any starting and ending configurations of n disks on three pegs that are more than $2^n - 1$ moves apart, under Lucas's original rules?

5. Let Q_n be the minimum number of moves needed to transfer a tower of n disks from A to B if all moves must be clockwise – that is, from A to B, or from B to the other peg, or from the other peg to A. Also let R_n be the minimum number of moves needed to go from B back to A under this restriction. Prove that
$$Q_n = \begin{cases} 0 & n = 0 \\ 2R_{n-1} + 1 & n > 0 \end{cases}$$
and
$$R_n = \begin{cases} 0 & n = 0 \\ Q_n + Q_{n-1} + 1 & n > 0. \end{cases}$$

6. A **Double Tower of Hanoi** contains $2n$ disks of n different sizes, two of each size. As usual, we're required to move only one disk at a time, without putting a larger one over a smaller one.
   (a) How many moves does it take to transfer a double tower from one peg to another, if disks of equal size are indistinguishable from each other?
   (b) What if we are required to reproduce the original top-to-bottom order of all the equal-size disks in the final arrangement?

7. Let's generalize exercise (6a) even further, by assuming that there are m different sizes of disks and exactly m_k disks of size k. Determine $A(m_1, \ldots, m_n)$, the minimum number of moves needed to transfer a tower when equal-size disks are considered to be indistinguishable.

8. If W_n is the minimum number of moves needed to transfer a tower of n disks from one peg to another when there are four pegs instead of three, show that
$$W_{n(n+1)/2} \leq 2W_{n(n-1)/2} + T_n, \quad \text{for} n > 0.$$
(Here $T_n = 2^n - 1$ is the ordinary three-peg number.) Use this to find a closed form $f(n)$ such that $W_{n(n-1)/2} \leq f(n)$ for all $n \geq 0$.

9. Solve infinitely many cases of the four-peg Tower of Hanoi problem by proving that equality holds in the relation of above exercise.

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