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Concrete Mathematics - MA 201
Problem Sheet - 6

1. Let n be an integer and x be real. Then prove that
   - $\lfloor x \rfloor = n \iff n \leq x < n+1$
   - $\lceil x \rceil = n \iff n-1 < x \leq x$
   - $\lfloor x \rfloor = n \iff x-1 < n \leq x$
   - $\lceil x \rceil = n \iff x \leq n < x+1$
   - $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.
2. Prove that $\lfloor x+n \rfloor = \lfloor x \rfloor + n$, for any integer n.
   Is $\lfloor x+n \rfloor = \lfloor x \rfloor + n$ true, for an arbitrary real n?
3. Prove that for any real numbers x and y, $\lfloor x+y \rfloor$ is either $\lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor + 1$. In general, $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.
4. Prove that $\lceil \lg(n+1) \rceil$ bits are needed to express n in binary, for all $n > 0$.
5. What is $\lceil \lfloor x \rfloor \rceil$ for any real number x?
6. Say true or false with justification : All expressions with an innermost $\lfloor x \rfloor$ surrounded by any number of floors or ceilings are same.
7. Prove that $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$ for any real $x \geq 0$.
8. Prove that
   $$\lfloor \frac{x+m}{n} \rfloor = \lfloor \frac{\lfloor x \rfloor + m}{n} \rfloor$$
   and
   $$\lceil \frac{x+m}{n} \rceil = \lceil \frac{\lceil x \rceil + m}{n} \rceil$$
   if m and n are integers and the denominator n is positive.
9. Prove or disprove the statement
   $$\lceil \sqrt{\lfloor x \rfloor} \rceil = \lceil \sqrt{x} \rceil \quad \text{for any real } x \geq 0. \tag{1}$$
   Does the assertion (1) work for $x = \pi$, $x = e$ and $x = \frac{1+\sqrt{5}}{2}$?
10. Prove the following:
    - $[\alpha, \beta]$ $(\alpha \leq \beta)$ contains $\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$ integers,
    - $(\alpha, \beta]$ $(\alpha \leq \beta)$ contains exactly $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$ integers,
    - (α, β) $(\alpha < \beta)$ contains exactly $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$ integers.
11. Find Spec($\frac{1}{2}$), Spec($\sqrt{2}$) and Spec($2+\sqrt{2}$).
12. In Josephus problem, we represented an arbitrary positive number n in the form $n = 2^m + \ell$, where $0 \leq \ell < 2^m$. Give explicit formulas for ℓ and m as functions of n, using floor and / or ceiling brackets.

13. What is a formula for the nearest integer to a given real number x? In case of ties, when x is exactly halfway that rounds

    (a) up, that is, to $\lceil x \rceil$

    (b) down, that is, to $\lfloor x \rfloor$.

14. Evaluate $\lfloor \lfloor m\alpha \rfloor n/\alpha \rfloor$, when m and n are positive integers and α is an irrational number greater than n.

15. Find a necessary and sufficient condition that For example, $\lfloor nx \rfloor \neq n\lfloor x \rfloor$ when n is a positive integer.

16. Prove the **Dirichlet box principle** :

    If n objects are put into m boxes, some box must contain $\geq \lceil n/m \rceil$ objects, and some box must contain $\leq \lfloor n/m \rfloor$.

17. Can something be said about $\lfloor f(x) \rfloor$ when $f(x)$ is a continuous monotonically decreasing function that takes integer values only when x is an integer?

18. Show that the expression

    $$\lceil \frac{2x+1}{2} \rceil - \lceil \frac{2x+1}{4} \rceil + \lfloor \frac{2x+1}{4} \rfloor$$

    is always either $\lfloor x \rfloor$ or $\lceil x \rceil$. In what circumstances does each case arise?

19. Let α and β be positive real numbers. Prove that $Spec(\alpha)$ and $Spec(\beta)$ partition the positive integers if and only if α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

20. Find a necessary and sufficient condition on the real number $b > 1$ such that $\lfloor \log_b x \rfloor = \lfloor \log_b \lfloor x \rfloor \rfloor$ for all real $x \geq 1$.

21. Find the sum of all multiples of x in the closed interval $[\alpha, \beta]$, when $x > 0$.

22. How many of the numbers 2^m, for $0 \leq m \leq M$, have leading digit 1 in decimal notation?

23. Evaluate the sums

    $$S_n = \sum_{k \geq 1} \lfloor \frac{n}{2^k} + \frac{1}{2} \rfloor$$

    and

    $$T_n = \sum_{k \geq k} 2^k \lfloor \frac{n}{2^k} + \frac{1}{2} \rfloor^2.$$

24. Show that $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$ is the nth element of the sequence

    $$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6 \cdots.$$

    The sequence has exactly k occurrences of 'k', for $k > 1$.

25. Prove that for any real x and positive integer m,

    $$\lfloor \frac{\lfloor x \rfloor}{m} \rfloor = \lfloor \frac{x}{m} \rfloor.$$

26. Prove that for any real x,

    (a) $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$

    (b) $\lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = \lfloor 3x \rfloor$

(c) $\lfloor x \rfloor + \lfloor x + \frac{1}{m} \rfloor + \lfloor x + \frac{2}{m} \rfloor + \cdots + \lfloor x + \frac{m-1}{m} \rfloor = \lfloor mx \rfloor$.

27. Prove that spectrum of $\sqrt{2}$ contains infinitely many powers of 2.

    That is, prove that there are infinitely many integers $n \geq 1$ such that $\lfloor n\sqrt{2} \rfloor = 2^k$ for some $k > 0$.

28. Solve the recurrence

$$\begin{aligned} X_n &= n, & \text{for } 0 \leq n < m, \\ X_n &= X_{n-m} + 1, & \text{for } n \geq m. \end{aligned}$$

29. Solve the recurrence

$$\begin{aligned} a_0 &= 1, \\ a_n &= a_{n-1} + \lfloor \sqrt{a_{n-1}} \rfloor, \quad \text{for } n > 0. \end{aligned}$$

30. Let α and β be positive real numbers. We have proved that $Spec(\alpha)$ and $Spec(\beta)$ partition the positive integers if and only if α and β are irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

    This establishes an **interesting relation** between the two multisets $Spec(\alpha)$ and $Spec(\alpha/(\alpha-1))$, when α is any irrational > 1, because and $\frac{1}{\alpha} + \frac{\alpha-1}{\alpha} = 1$.

    Find (and prove) an **interesting relation** between the two multisets $Spec(\alpha)$ and $Spec(\alpha/(\alpha+1))$, where α is any positive real number.

31. Prove or disprove : $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$.

32. Let $\|x\| = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$ denote the distance from x to the nearest integer.

    What is the value of

$$\sum_k 2^k \|x/2^k\|^2?$$

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