Sequences

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Overview

We shall study a special class of functions (called **sequences**) whose domain is the set of all natural numbers and range a set of real numbers.

Analysis is based on the notion of a limit, a concept that can be defined in terms of sequences. Moreover, elementary functions, such as trigonometric, exponential, and logarithm functions and many algebraic functions, can be approximated by using sequences.

With modern computers, such approximations can be made accurate enough for most practical purposes.

We shall discuss sequences of real numbers in a couple of lectures.

A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the **terms** of the sequence.

The integer n is called the **index** of a_n, and indicates where a_n occurs in the list. We can think of the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

as a function that sends 1 to a_1, 2 to a_2, 3 to a_3, and in general, **the function sends the positive integer n to the nth term a_n**. This leads to the formal definition of a sequence.

Definition 1 (Infinite Sequence).

An infinite sequence *of numbers is a function whose domain is the set of positive integers.*

Example 2.

The function associated to the sequence

$$2, 4, 6, 8, 10, 12, \dots, 2n, \dots$$

sends 1 *to* $a_1 = 2$, 2 *to* $a_2 = 4$, *and so on. The general behavior of this sequence is described by the formula*

$$a_n = 2n.$$

Infinite Sequence

We can equally well make **the domain, the integers larger than a given number** n_0, and we allow sequences of this type also.

The sequence

$$12, 14, 16, 18, 20, 22, \ldots$$

is described by the formula $a_n = 10 + 2n$. It can also described by the simpler formula $b_n = 2n$, where the index n starts at 6 and increases.

To allow such simpler formulas, we let the first index of the sequence be any integer. In the sequence above, $\{a_n\}$ starts with a_1 while $\{b_n\}$ starts with b_6.

Order is important. The sequence $1, 2, 3, 4, \ldots$ is not the same as the sequence $2, 1, 3, 4, \ldots$.

Examples of Sequences

Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n} \qquad c_n = \frac{n-1}{n}$$
$$b_n = (-1)^{n+1}\frac{1}{n} \qquad d_n = (-1)^{n+1}$$

or by listing terms,

$$\{a_n\} = \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$
$$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}$$
$$\{c_n\} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}$$
$$\{d_n\} = \left\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\right\}.$$

We also sometimes write

$$\{a_n\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}.$$

What is the difference between a set and a sequence?

Question : What is the difference between a set and a sequence?

Sequence always has a definite order of elements.

A set is a (well-defined) collection of distinct elements (that contains no duplicate elements).

$\{1, -1, 1, -1, 1, -1, 1, \ldots\}$ is a sequence whose elements are from the set $\{-1, 1\}$.

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \ldots \frac{1}{n}, \ldots \right\}$$

whose terms approach 0 as n gets large, and in the sequence

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1 - \frac{1}{n}, \ldots \right\}$$

whose terms approach 1.

Convergence and Divergence

On the other hand, sequences like

$$\left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}$$

have terms that get larger than any number as n increases, and sequences like

$$\left\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\right\}$$

bounce back and forth between 1 and -1, never converging to a single value.

Converges, Diverges, Limit

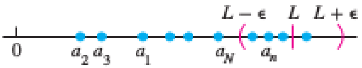
Definition 3 (Converges, Diverges, Limit).

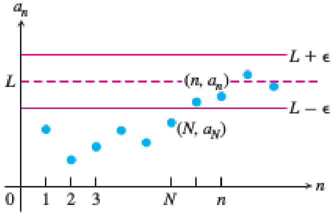
The sequence $\{a_n\}$ converges *to the number* L *if for every positive number* ε *there corresponds a positive integer* N *such that for all* n,

$$n > N \implies |a_n - L| < \varepsilon.$$

That is,

$$L - \varepsilon < a_n < L + \varepsilon \quad \textit{for all } n > N.$$





If no such number L exists, we say that $\{a_n\}$ **diverges**.

Definition of Convergence

In some textbooks, the following definition is used for convergence.

Definition 4.

The sequence $\{a_n\}$ converges *to the number* L *if for every positive number* ε *there corresponds a positive integer* N *such that for all* n,

$$n \geq N \implies |a_n - L| < \varepsilon.$$

The only difference is that we have $n \geq N$ instead of $n > N$. When $n > N$ we mean, we are concerned about the terms of the sequence from $N + 1$ onwards ; whereas $n \geq N$ we mean, we are concerned about the terms of the sequence from N onwards. This does not affect the choice of N. You will understand that the convergence/divergence of a sequence which does not depend on first finitely many terms. A finite term can be a number (distance) from NITK to any city in India, in milimeters.

The shortest math joke: Let epsilon be < 0.

Let $\{a_n\}$ be a sequence converging to a real number, say L.

Now we apply the definition of convergence of $\{a_n\}$ to L for ε. We will get a natural number, say N_1, such that

$$n > N_1 \implies |a_n - L| < \varepsilon.$$

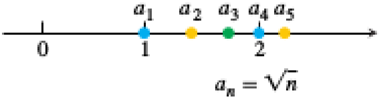
Again we apply the definition of convergence of $\{a_n\}$ to L for $\varepsilon/99$. We will get a natural number, say N_2 such that

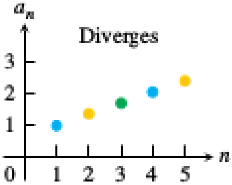
$$n > N_2 \implies |a_n - L| < \varepsilon.$$

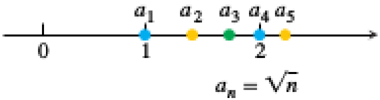
Caution : We should not think that N_2 is always more that $99N_1$. However, N_2 is more than N_1.

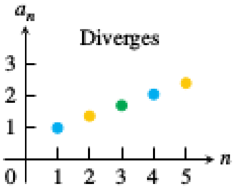
The following figure shows **two ways to represent sequences graphically.**

- The first marks the first few points from $a_1, a_2, a_3, \ldots, a_n, \ldots$ on the real axis.
- The second method shows the graph of the function defining the sequence. The function is defined only on integer inputs, and the graph consits of some points in the xy-plane, located at $(1, a_1), (2, a_2), \ldots, (n, a_n), \ldots$.



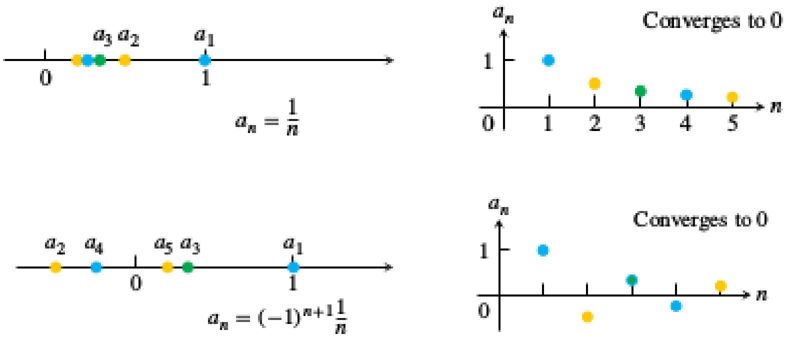






Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

Representing Sequences Graphically



Sequences can be represented as points **on the real line** or as points **in the plane** where the horizontal axis n is the index number of the term and the vertical axis a_n is its value.

Visualizing Sequences

As discussed above, there are two ways to visualize a sequence of real numbers.

- **In real line :** We can have the terms of the sequence $\{a_n\}$ on the real line.
- **In the plane :** Every sequence (of real numbers) is a function (whose domain is the set of positive integers) and range (image) is in the set of real numbers.

Which is good for better understanding?

Let us consider the constant sequence $\{10\}_{n=1}^{\infty}$.

- **On the real line :** We get the only one point, 10, on the real line. Given $\varepsilon > 0$, we have an interval $(10 - \varepsilon, 10 + \varepsilon)$ which contains every element of the sequence $\{10\}_{n=1}^{\infty}$, hence any natural number N will satisfy the following :

$$n > N \implies |a_n - L| = |10 - 10| < \varepsilon.$$

- **In the plane :** We get points on the line $y = 10$ defined over the set of positive integers. Given $\varepsilon > 0$, we get two lines $y = 10 - \varepsilon$ and $y = 10 + \varepsilon$. As every dot (n, a_n) lies between $10 + \varepsilon$ and $10 - \varepsilon$, we can choose any natural number N which will satisfy the following :

$$n > N \implies |a_n - L| = |10 - 10| < \varepsilon.$$

Common Mistake

Some students may write the following **incomplete line**

$$L - \varepsilon < a_n < L + \varepsilon$$

for the definition of convergence which is not correct.

Given a seqence $\{a_n\}$, we guess some number L where the sequence might converge.

To prove our guess L is correct, we start with $\varepsilon > 0$, we find a positive integer N such that when $n > N$ we must prove $|a_n - L| < \varepsilon$.

That is, given $\varepsilon > 0$, we shall find a positive integer N such that

$$n > N \implies |a_n - L| < \varepsilon.$$

Uniqueness of Limits

Theorem 5 (Uniqueness of Limits).

Every convergence sequence has a unique point.

That is, if $\{a_n\}$ *is a sequence of real numbers and, if* L_1 *and* L_2 *are numbers such that*

$$a_n \to L_1 \quad \text{and} \quad a_n \to L_2,$$

then

$$L_1 = L_2.$$

In other words, a sequence cannot converge to more than one limit.

Outline of the proof

We would like to show that $|L_1 - L_2| < \varepsilon$ for every $\varepsilon > 0$.

Why is this useful ?
For every $\varepsilon > 0$, if we have $|L_1 - L_2| < \varepsilon$, then $|L_1 - L_2| = 0$, hence $L_1 = L_2$. [Reason : Suppose $L_1 \neq L_2$, for $\varepsilon = \frac{|L_1 - L_2|}{2}$, we get that $\varepsilon = \frac{|L_1 - L_2|}{2} < |L_1 - L_2|$ which contradicts to $|L_1 - L_2| < \varepsilon$, for every $\varepsilon > 0$.]

How to show $|L_1 - L_2| < \varepsilon$ for every $\varepsilon > 0$?
We have

$$\begin{aligned} |L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\ &\leq |a_n - L_1| + |a_n - L_2|. \end{aligned}$$

In order to show $|L_1 - L_2| < \varepsilon$, we shall show that $|a_n - L_1| < \varepsilon/2$ and $|a_n - L_2| < \varepsilon/2$ by applying the definition of convergence for $\varepsilon/2$.

Proof

Let $\{a_n\}$ be a sequence of real numbers.

Suppose that $\{a_n\}$ converges to **two distinct numbers** L_1 and L_2.

Let $\varepsilon > 0$ be given.

Since $a_n \to L_1$, there is a positive integer N_1 such that

$$|a_n - L_1| < \varepsilon/2 \quad \text{for all } n > N_1.$$

Similarly, as $a_n \to L_2$, there is a positive integer N_2 such that

$$|a_n - L_2| < \varepsilon/2 \quad \text{for all } n > N_2.$$

Proof (contd...)

Now, for $N > \max\{N_1, N_2\}$,

$$\begin{aligned} |L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\ &\leq |a_n - L_1| + |a_n - L_2| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence

$$0 < |L_1 - L_2| < \varepsilon, \quad \text{for every } \varepsilon > 0. \tag{1}$$

But if ε is chosen to be $\frac{|L_1 - L_2|}{2}$, we get a contradiciton with (1).

Thus, the sequence cannot converge to two limits.

Note : In the above proof, we considered a partition of ε as $\frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. If we consider the partition as $\frac{\varepsilon}{100} + \frac{99\varepsilon}{100}$, we may get the some natural numbers M_1 and M_2 (may be different from N_1 and N_2) satisfying inequalities.

Converges, Diverges, Limit

The definition says that if we go far enough out in the sequence, by taking the index n to be larger then some value N, the difference a_n and the limit of the sequence becomes less than **any preselected number** $\varepsilon > 0$.

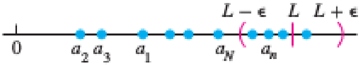
The definition is very similar to the definition of the limit of a function $f(x)$ **as** x **tends to** ∞**.** We will exploit this connection to calculate limits of sequences.

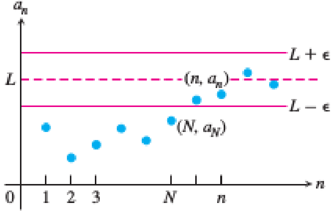
Convergence and Divergence

If $\{a_n\}$ converges to L, we write

$$\lim_{n\to\infty} a_n = L, \quad \text{or simply} \quad a_n \to L,$$

and call the **limit** of the sequence.





In the representation of a sequence as points in the plane, $a_n \to L$ if $y = L$ is a **horizontal asymptote** of the sequence of points $\{(n, a_n)\}$. In this figure, all the a_n's after a_N lie within ε of L.

Exercise 6.

A formula for the nth term a_n of a sequence $\{a_n\}$ is given. Find the values of $a_1, a_2, a_3,$ and a_4.

1. $a_n = \frac{1-n}{n^2}$
2. $a_n = 2 + (-1)^n$
3. $a_n = \frac{2^n}{2^{n+1}}$

Note that when the index set for n is not explicitly given, it is assumed that the index set is the set of natural numbers.

Exercise 7.

The first term or two of a sequence along with a recursion formula for the remaining terms are given. Write out the first ten terms of the sequence.

1. $a_1 = 1, a_{n+1} = a_n + (1/2^n)$
2. $a_1 = -2, a_{n+1} = na_n/(n+1)$
3. $a_1 = 2, a_2 = -1, a_{n+2} = a_{n+1}/a_n$

Exercise 8 (Finding a Sequence's Formula).

Find a formula for the nth term of the sequence.

1. *The sequence* $-1, 1, -1, 1, -1, 1, \ldots$
2. *The sequence* $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$
3. *The sequence* $-3, -2, -1, 0, 1, \ldots$
4. *The sequence* $0, 1, 1, 2, 2, 3, 3, 4, \ldots$
5. *The sequence* $1, 1, \ldots, 1$*(10 times)*$, 2, 2, \ldots, 2$*(10 times)*$, \ldots$
   (Each positive integer is repeated 10 *times.)*
6. *The sequence* $1, 2, 2$*(2 times)*$, 3, 3, 3$*(3 times)*$, \ldots$
   (Each positive integer n is repeated n times.)

Solution for Exercise 8

1. $(-1)^n, \quad n \geq 1$
2. $\frac{(-1)^{n+1}}{n^2}, \quad n \geq 1$
3. $n - 4, \quad n \geq 1$
4. $\lfloor \frac{n}{2} \rfloor, \quad n \geq 1$
5. $\lceil \frac{n}{10} \rceil, \quad n \geq 1$
6. $\left\lfloor \frac{1+\sqrt{1+8n}}{2} \right\rfloor, \quad n \geq 1$

Sequence's formula is not unique.

For example, one can find the following sequence's formula

$$a_n = \frac{4}{15}n^5 - \frac{14}{3}n^4 + \frac{92}{3}n^3 - \frac{280}{3}n^2 + \frac{1936}{15}n - 63$$

for the sequence $\{-1, 1, -1, 1, -1, 1, \dots\}$.

For $n = 1, 2, 3, 4, 5, 6$, we get $-1, 1, -1, 1, -1, 1$ respectively. But $a_7 = 63$ (not -1).

A simple formula is $a_n = (-1)^n$ for $n \geq 1$.

If we know the first finitely many terms of a sequence, say $\{a_n\}$, every one in world can give a sequence's formula for the sequence $\{-1, 1, -1, 1, -1, 1, \dots\}$ which satisfies $a_n = (-1)^n, 1 \leq n \leq 6$, and a_7 can be any number of his/her choice.

Sequence's formula is not unique.

Two students were asked to write an nth term for the sequence

$$1, 16, 81, 256, \ldots$$

and to write 5th term of the sequence. One student gave the nth term as

$$a_n = n^4.$$

The other student, who did not recognize this simple law of formation, wrote

$$a_n = 10n^3 - 35n^2 + 50n - 24.$$

Which student gave the correct 5th term?

Using $\varepsilon - N$ definition, prove the convergence of sequences.

Example 9 (Applying the $\varepsilon - N$ Definition of Convergence).

Show that

1. $\lim_{n\to\infty} \frac{1}{n} = 0$
2. $\lim_{n\to\infty} k = k$ *(any constant* k*).*

Solution

1. Let $\varepsilon > 0$ be given. We must show that there exists a positive integer N such that for all n,
$$n > N \implies \left|\frac{1}{n} - 0\right| < \varepsilon.$$
This implication will hold if $\frac{1}{n} < \varepsilon$ or $n > 1/\varepsilon$. If N is any integer greater than $1/\varepsilon$ (or, choose $N = \lceil 1/\varepsilon \rceil$), the implication will hold for all $n > N$. That is, $n > N \implies |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$. This proves that $\lim_{n\to\infty}(1/n) = 0$.
2. Let $\varepsilon > 0$ be given. We must show that there exists a positive integer N such that for all n,
$$n > N \implies |k - k| < \varepsilon.$$
Since $k - k = 0$, we can choose any positive integer for N (or, choose $N = 10$ or $N = 1000$ billion) and the implication will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant k.

Bounded Sequence

Definition 10.

A sequence $\{a_n\}$ *is said to be* bounded *if there are real numbers* m_1 *and* m_2 *such that*

$$m_1 \leq a_n \leq m_2 \quad \text{for all } n.$$

One can use the following **equivalent statement for bounded**: $\{a_n\}$ is bounded if there is a positive number M such that

$$|a_n| \leq M, \quad \text{for all } n.$$

Theorem 11.

Every convergent sequence is bounded.

Proof

Let $\{a_n\}$ be a sequence converging to a limit L.

Let $\varepsilon > 0$ be a fixed number, say 100. Since $a_n \to L$, there is a positive integer N such that

$$|a_n - L| < 100, \quad \text{for all } n > N.$$

That is,

$$L - 100 < a_n < L + 100, \quad \text{for all } n > N.$$

Note that for $n \geq N + 1$, all a_n's will lie in the interval $(L - 100, L + 100)$.

What about the terms $a_1, a_2, \ldots, a_N$?
Some of them may lie in the interval $(L - \varepsilon, L + \varepsilon)$, or, some of them may not lie in the interval.

Let $m_1 = \min\{a_1, a_2, \ldots, a_N, L - \varepsilon\}$ and $m_2 = \max\{a_1, a_2, \ldots, a_N, L + \varepsilon\}$. We get that for all n,

$$m_1 \leq a_n \leq m_2.$$

Thus $\{a_n\}$ is bounded. This completes the proof.

What is special about 100 in the above proof?

We could have used 1 in place of 100 and apply convergence of $\{a_n\}$ to L for $\varepsilon = 1$ to get some positive integer, say M.

Hence $n > M \implies |a_n - L| < 1$ and proceed.

Observations : We proved that every convergent sequence is bounded. What about the converse of the above statement? It is not true. That is, a bounded sequence is not necessarily convergent. For example, the seqence $\{1, 2, 3, 4, 1, 2, 3, 4, \ldots\}$ is bounded, but not convergent.

Boundedness is a necessary condition for the convergence of a sequence but not a sufficient condition. That is, if a sequence is not bounded, then it cannot converge. For example, $\{n!\}$ is not convergent because it is not bounded.

Example 12.

Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ *diverges.*

Solution : Suppose the sequence converges to some number L. By choosing $\varepsilon = 1/2$ in the definition of the limit, all terms a_n of the sequence with index n larger than some N must lie within $\varepsilon = 1/2$ of L. Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\varepsilon = 1/2$ of L. It follows that $|L - 1| < 1/2$, or equivalently, $1/2 < L < 3/2$. Likewise, the number -1 appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L - (-1)| < 1/2$, or equivalently, $-3/2 < L < -1/2$. But the number L cannot lie in both of the intervals $(1/2, 3/2)$ and $(-3/2, -1/2)$ because they have no overlap. Therefore, no such limit L exists and so the sequence diverges. **Note that the same argument works for any positive number** ε **smaller than** 1**, not just** $1/2$**.**

Divergent Sequences

The sequence $\{\sqrt{n}\}$ also diverges, but for a different reason. As n increases, its terms become larger than any fixed number. We describe the behavior of this sequence by writing $\lim_{n\to\infty} \sqrt{n} = \infty$.

1. In writing infinity as the limit of a sequence, we are not saying that the differences between the terms a_n and ∞ become small as n increases.
2. Nor are we asserting that there is some number infinity that the sequence approaches.
3. We are merely using a notation that captures the idea that a_n eventually gets and stays larger than any n gets large.

Definition 13 (Diverges to Infinity).

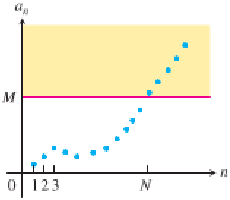
The sequence $\{a_n\}$ diverges to infinity *if for every number* M *there is an integer* N *such that for all* n *larger than* N, $a_n > M$. *If this condition holds we write*

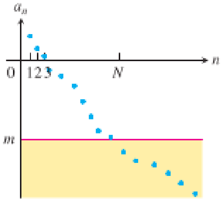
$$\lim_{n\to\infty} a_n = \infty \quad \textit{or} \quad a_n \to \infty.$$

Similarly if for every number m *there is an integer* N *such that for all* $n > N$ *we have* $a_n < m$, *then we say* $\{a_n\}$ diverges to negative infinity *and write*

$$\lim_{n\to\infty} a_n = -\infty \quad \textit{or} \quad a_n \to -\infty.$$

Diverges to $+\infty$ or $-\infty$





The sequence diverges to ∞ because no matter what number M is chosen, the terms of the sequence after some index N all lie in the yellow band above M.

The sequence diverges to $-\infty$ because all terms after some index N lie below any chosen number m.

Diverges to Infinity

Example 14.

A sequence may diverge without diverging to infinity or negative infinity. The sequences

$$\{1, -2, 3, -4, 5, -6, 7, -8, \ldots\}$$

and

$$\{1, 0, 2, 0, 3, 0, \ldots\}$$

are also examples of such divergence.

Calculating Limits of Sequences

If we always had to use the formal definition of the limit of a sequence, calculating with ε's and N's, then **computing limits of sequences would be a formidable task.**

Fortunately we can derive a few basic examples, and then use these to quickly analyze the limits of many more sequences.

We will need to understand how to combine and compare sequences. Since sequences are functions with domain restricted to the positive integers, it is not too surprising that the theorems on limits of functions have versions for sequences.

Theorem 15.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. Sum Rule : $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. Difference Rule : $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. Product Rule : $\lim_{n\to\infty}(a_n.b_n) = A.B$
4. Constant Multiple Rule : $\lim_{n\to\infty}(k.b_n) = k.B$ *(any number k)*
5. Quotient Rule : $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ *if* $b_n \neq 0$ *for all* n *and* $B \neq 0$

Proof (Sum Rule)

Let $\varepsilon > 0$ be given.

Rough Work (not necessarily to be written in exam) :
We have to find a positive integer N such that
$|(a_n + b_n) - (A + B)| < \varepsilon$ for all $n > N$.
From the triangle inequality, we have
$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B|$.
Hence we can apply the defintion for $\varepsilon/2$, for the sequences $\{a_n\}$ and $\{b_n\}$ converging to A and B respectively.

Since $a_n \to A$ and $b_n \to B$, there are positive integers N_1 and N_2 respectively, such that

$$|a_n - A| < \frac{\varepsilon}{2} \quad \text{for all } n > N_1$$

and

$$|b_n - B| < \frac{\varepsilon}{2} \quad \text{for all } n > N_2.$$

Proof (contd...)

Thus for all $n > N = \max\{N_1, N_2\}$,

$$\begin{aligned} |(a_n + b_n) - (A + B)| &\leq |a_n - A| + |b_n - B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{a_n + b_n\}$ converges to $A + B$.

Proof (Difference Rule)

Let $\varepsilon > 0$ be given.

Rough Work (not necessarily to be written in exam) :
We have to find a positive integer N such that
$|(a_n - b_n) - (A - B)| < \varepsilon$ for all $n > N$.
From the triangle inequality, we have
$|(a_n - b_n) - (A - B)| \leq |a_n - A| + |b_n - B|$.
Hence we can apply the defintion for $\varepsilon/2$, for the sequences $\{a_n\}$ and $\{b_n\}$ converging to A and B respectively.

Since $a_n \to A$ and $b_n \to B$, there are positive integers N_1 and N_2 respectively, such that

$$|a_n - A| < \frac{\varepsilon}{2} \quad \text{for all } n > N_1$$

and

$$|b_n - B| < \frac{\varepsilon}{2} \quad \text{for all } n > N_2.$$

Proof (contd...)

Thus for all $n > N = \max\{N_1, N_2\}$,

$$\begin{aligned} |(a_n - b_n) - (A - B)| &\leq |a_n - A| + |b_n - B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{a_n - b_n\}$ converges to $A - B$.

Proof (Product Rule)

Let $\varepsilon > 0$ be given.

Rough Work (not necessarily to be written in exam) :
We have to find a positive integer N such that $|a_n b_n - AB| < \varepsilon$ for all $n > N$.
From the triangle inequality, we have
$|a_n b_n - AB| = |a_n b_n - a_n B + a_n B - AB| \leq |a_n|\ |b_n - B| + |a_n - A|\ |B|$.
Hence one may want to apply the defintion for ε as $\frac{\varepsilon}{2|B|}$ and $\frac{\varepsilon}{2|a_n|}$, for the sequences $\{a_n\}$ and $\{b_n\}$ converging to A and B respectively.
Here there are some problems. The number $\frac{\varepsilon}{2|a_n|}$ is depending on n ;
when B is zero, what happens to the number $\frac{\varepsilon}{2|B|}$.
So, **we are using boundedness of the convergent sequence** $\{a_n\}$ **;**
and **we are discussing a separate case when** $B = 0$**.**

Since every convergent sequence is bounded, for the sequence $\{a_n\}$, there exists a positive number C such that

$$|a_n| \leq C, \quad \text{for all } n.$$

Proof (contd...)

Case 1 : B **is non-zero**

Since $a_n \to A$ and $b_n \to B$, there are positive integers N_1 and N_2 respectively, such that

$$|a_n - A| < \frac{\varepsilon}{2|B|} \quad \text{for all } n > N_1$$

and

$$|b_n - B| < \frac{\varepsilon}{2C} \quad \text{for all } n > N_2.$$

Thus for all $n > N = \max\{N_1, N_2\}$,

$$\begin{aligned} |a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\ &\leq |a_n|\,|b_n - B| + |a_n - A|\,|B| \\ &\leq C\,|b_n - B| + |a_n - A|\,|B| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{a_n b_n\}$ converges to AB when $B \neq 0$.

Case 2 : B **is zero**

Rough Work (not necessarily to be written in exam) :
We have to find a positive integer N such that $|a_n b_n - 0| < \varepsilon$ for all $n > N$.
From the triangle inequality, we have $|a_n b_n - 0| = |a_n|\ |b_n - 0|$.
Hence one can apply the "$\varepsilon - N$" defintion for the sequence $\{b_n\}$ converging to 0.

Since $b_n \to B$, there is a positive integer N_1 such that

$$|b_n - 0| < \frac{\varepsilon}{C} \quad \text{for all } n > N_1.$$

Thus for all $n > N_1$,

$$\begin{aligned} |a_n b_n - 0| &= |a_n|\ |b_n - 0| \\ &\leq C|b_n - 0| \\ &< \varepsilon. \end{aligned}$$

Hence $\{a_n b_n\}$ converges to 0 when $B = 0$.

This completes the proof.

Proof (Constant Multiple Rule)

Let $\varepsilon > 0$ be given.

If k is zero, $\{ka_n\}$ is a constant sequence, converging to the constant, which is 0 here. So, let's assume that k is non-zero.

Rough Work (not necessarily to be written in exam) :
We have to find a positive integer N such that $|ka_n - kA| < \varepsilon$ for all $n > N$.
From the triangle inequality, we have $|ka_n - kA| = |k|\ |a_n - A|$.
Hence one can apply the "$\varepsilon - N$" defintion for the sequence $\{a_n\}$ converging to A, for $\varepsilon/|k|$.

Since $a_n \to A$, there is a positive integer N such that

$$|a_n - A| < \frac{\varepsilon}{|k|} \quad \text{for all } n > N.$$

Thus for all $n > N$,

$$\begin{aligned} |ka_n - kA| &= |k|\ |a_n - A| \\ &< \varepsilon. \end{aligned}$$

Hence $\{ka_n\}$ converges to kA.

To prove the quotient rule, we use the following lemma.

Lemma 16.

If $\lim_{n\to\infty} b_n = B \neq 0$*, then there is a positive integer* N *such that*

$$|b_n| > \frac{|B|}{2}, \quad \textit{for all } n > N.$$

Proof of the lemma :

Since $b_n \to B$ and $\varepsilon = \frac{|B|}{2}$, there exists a positive integer N such that

$$|b_n - B| < \frac{|B|}{2}, \quad \text{for all } n > N.$$

This implies that

$$|B| - |b_n| \leq |b_n - B| < \frac{|B|}{2}, \quad \text{for all } n > N$$

hence,

$$|b_n| > \frac{|B|}{2}, \quad \text{for all } n > N.$$

Proof (contd...)

Let us apply the Lemma to prove the quotient rule.

Let $\varepsilon > 0$ be given.

Rough Work (not necessarily to be written in exam) :
We have to find a positive integer N such that
$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \frac{|Ba_n - Ab_n|}{|B|.|b_n|} = \frac{|Ba_n - BA - Ab_n + AB|}{|B|.|b_n|} \leq \frac{|B|.|a_n - A| + |A|.|b_n - B|}{|B|.|b_n|} \leq \frac{|a_n - A|}{|b_n|} + \frac{|A|.|b_n - B|}{|B|.|b_n|}$
Using lemma, there exists a positive integer N such that $|b_n| > \frac{|B|}{2}$, for all $n > N$.
We get $\left|\frac{a_n}{b_n} - \frac{A}{B}\right| < \frac{2}{|B|}|a_n - A| + \frac{2|A|.|b_n - B|}{|B|^2}$,
Hence we can apply the defintion for ε as $\frac{|B|\varepsilon}{4}$ and $\frac{|B|^2\varepsilon}{4|A|}$, for the sequences $\{a_n\}$ and $\{b_n\}$ converging to A and B respectively. The case for $A = 0$, should be discussed separetely.

<u>Case 1 :</u> A is non-zero

Since $a_n \to A$ and $b_n \to B$, there are positive integers N_1 and N_2 respectively, such that

$$|a_n - A| < \frac{|B|\varepsilon}{4} \quad \text{for all } n > N_1$$

and
$$|b_n - B| < \frac{|B|^2\varepsilon}{4|A|} \quad \text{for all } n > N_2.$$
By lemma, there exists a positive integer N_3 such that
$$|b_n| > \frac{|B|}{2}, \quad \text{for all } n > N_3.$$

Thus for all $n > N = \max\{N_1, N_2, N_3\}$,
$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{A}{B}\right| &= \frac{|Ba_n - Ab_n|}{|B|.|b_n|} = \frac{|Ba_n - BA - Ab_n + AB|}{|B|.|b_n|}\\ &\leq \frac{|B|.|a_n - A| + |A|.|b_n - B|}{|B|.|b_n|} \leq \frac{|a_n - A|}{|b_n|} + \frac{|A|.|b_n - B|}{|B|.|b_n|}\\ &< \frac{2}{|B|}|a_n - A| + \frac{2|A|.|b_n - B|}{|B|^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $\{\frac{a_n}{b_n}\}$ converges to $\frac{A}{B}$ when $A \neq 0$.

<u>Case 2 :</u> A **is zero**

Rough Work (not necessarily to be written in exam) :
We have to find a positive integer N such that $\left|\frac{a_n}{b_n} - 0\right| < \varepsilon$ for all $n > N$.
From the triangle inequality, we have $\left|\frac{a_n}{b_n} - 0\right| = \frac{1}{|b_n|} \; |a_n - 0|$.
Hence one can apply the lemma for the sequence $\{b_n\}$ and the "$\varepsilon - N$" defintion for the sequence $\{a_n\}$ converging to 0.

By lemma, there exists a positive integer N_1 such that

$$|b_n| > \frac{|B|}{2}, \quad \text{for all } n > N_1.$$

Proof (contd...)

Since $a_n \to A$, there is a positive integer N_2 such that

$$|a_n - 0| < \frac{|B|}{2\varepsilon} \quad \text{for all } n > N_2.$$

Thus for all $n > N = \max\{N_1, N_2\}$,

$$\begin{aligned} \left|\frac{a_n}{b_n} - 0\right| &= \frac{1}{|b_n|}\,|a_n - 0| \\ &= \frac{2}{|B|}\,|a_n - 0| \\ &< \varepsilon. \end{aligned}$$

Hence $\left\{\frac{a_n}{b_n}\right\}$ converges to 0 when $A = 0$.

This completes the proof.

Applying Theorem for Limits of Sequences

Example 17.

1. $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1.\lim_{n\to\infty}\frac{1}{n} = -1.0 = 0$
2. $\lim_{n\to\infty}\frac{n-1}{n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$
3. $\lim_{n\to\infty}\frac{5}{n^2} = 5.\lim_{n\to\infty}\frac{1}{n}.\lim_{n\to\infty}\frac{1}{n} = 5.0.0 = 0$
4. $\lim_{n\to\infty}\frac{4-7n^6}{n^6+3} = \lim_{n\to\infty}\frac{(4/n^6)-7}{1+(3/n^6)} = \frac{0-7}{1+0} = -7$

Algebra of Limits

While the sum, difference, product and quotient (under appropriate conditions) of two convergent sequences is convergent, the converse may not be true.

That is, if $\{a_n \pm b_n\}$, $\{a_n b_n\}$ or $\{\frac{a_n}{b_n}\}$ is convergent, the component sequences $\{a_n\}$ and $\{b_n\}$ may not be convergent. However, both these sequences $\{a_n\}$ and $\{b_n\}$ shall behave alike.

Equivalently, existence of $\lim_{n\to\infty} (a_n \pm b_n)$ does not necessarily imply the existence of the two limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$. Analogously, for the product and quotient.

Algebra of Limits

For example, consider the sequnces $\{a_n\}$ and $\{b_n\}$ below:

- When $a_n = n^2$ and $b_n = -n^2$,
  $\{a_n + b_n\} \to 0$ and $\{\frac{a_n}{b_n}\} \to -1$, but both $\{a_n\}$ and $\{b_n\}$ are divergent.
- When $a_n = b_n = (-1)^n$,
  $\{a_n - b_n\} \to 0$, $\{\frac{a_n}{b_n}\} \to 1$ and $\{a_n b_n\} \to 1$, whereas both $\{a_n\}$ and $\{b_n\}$ oscillate finitely.
- If $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$, then
  $\{a_n + b_n\} \to 0$, $\{\frac{a_n}{b_n}\} \to -1$ and $\{a_n b_n\} \to -1$, while both $\{a_n\}$ and $\{b_n\}$ are not convergent (or oscillate finitely).

Algebra of Limits

If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges. But, when both $\{a_n\}$ and $\{b_n\}$ diverge, $\{a_n + b_n\}$ may converge, diverge or oscillate.

However, if $\{a_n\}$ converges and $\{b_n\}$ diverges, then nothing can be said about $\{a_n b_n\}$. That is, $\{a_n b_n\}$ may converge, diverge or oscillate.

It follows from Constant Multiple rule that "every non-zero multiple of a divergent sequence is divergent." That is, for any $c \neq 0$, if $\{a_n\}$ is divergent, then $\{ca_n\}$ must also be divergent.

For suppose, to the contrary, that $\{ca_n\}$ converges for some number $c \neq 0$. Then, taking $k = 1/c$ in the "Constant Multiple Rule", we see that the sequence

$$\left\{\frac{1}{c}.ca_n\right\} = \{a_n\}$$

converges. Thus, $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

Exercise 18.

1. *Give examples of sequences* $\{a_n\}$ *and* $\{b_n\}$ *such that*
   (a) $a_n \to +\infty$, $b_n \to -\infty$, *but* $\{a_n + b_n\}$ *converges.*
   (b) $a_n \to +\infty$, $b_n \to -\infty$, *but* $\{a_n + b_n\}$ *diverges to* $-\infty$.
   (c) $a_n \to +\infty$, $b_n \to -\infty$, *but* $\{a_n + b_n\}$ *oscillates.*
2. *Give examples of sequences* $\{a_n\}$ *and* $\{b_n\}$ *such that*
   (a) $a_n \to +\infty$, $\{b_n\}$ *converges, but* $\{a_n b_n\}$ *converges.*
   (b) $a_n \to +\infty$, $\{b_n\}$ *converges, but* $\{a_n b_n\}$ *diverges to* $+\infty$.
   (c) $a_n \to +\infty$, $\{b_n\}$ *converges, but* $\{a_n b_n\}$ *oscillates.*

Solution for Exercise 18

1. (a) $a_n = n$ and $b_n = -n$
   (b) $a_n = n$ and $b_n = -2n$
   (c) $a_n = n$ and $b_n = (-n + (-1)^n)$
2. (a) $a_n = n$ and $b_n = \frac{1}{n}$
   (b) $a_n = n$ and $b_n = 1$
   (c) $a_n = n$ and $b_n = \begin{cases} \frac{1}{n} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$

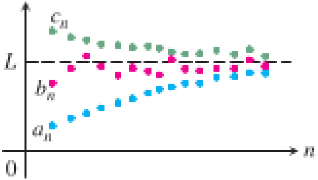
The next theorem is the sequence version of the Sandwich Theorem.

Theorem 19 (Sandwich / Squeeze Theorem for Sequences).

Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$ also.



Proof of Sandwich Theorem

Since $a_n \to L$ and $c_n \to L$, there are positive integers N_1 and N_2 respectively, such that

$$L - \varepsilon < a_n < L + \varepsilon \quad \text{for all } n > N_1$$

and

$$L - \varepsilon < c_n < L + \varepsilon \quad \text{for all } n > N_2.$$

Given that

$$a_n \leq b_n \leq c_n, \quad \text{for all } n > N.$$

Thus for all $n > N_3 = \max\{N_1, N_2, N\}$,

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \quad \text{for all } n > N.$$

Hence $\{b_n\}$ converges to L.

Sandwich Theorem

An immediate consequence of the Sandwich Theorem for sequences is that, if $|b_n| \leq c_n$ and $c_n \to 0$, then $b_n \to 0$ because $-c_n \leq b_n \leq c_n$.

We use this fact in the next example.

Example 20.

Since $1/n \to 0$,

(a) $\frac{\cos n}{n} \to 0$ *because* $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$

(b) $\frac{1}{2^n} \to 0$ *because* $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$

(c) $(-1)^n \frac{1}{n} \to 0$ *because* $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$.

An Application of Sequences - Bisection Method (Bolzano Method)

The **bisection method** is based on the Intermediate Value Theorem which states that if a function $f(x)$ is continuous over an interval $[a, b]$, then the function takes on every value between $f(a)$ and $f(b)$.

Suppose f is a continuous function defined on $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. By the Intermediate Value Theorem, there exists a number α in (a, b) with $f(\alpha) = 0$.

Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique. The method calls for a repeated halving of subintervals of $[a, b]$ and, at each step, locating the half containing α.

Algorithm

To begin, set $a_1 = a$ and $b_1 = b$, and let x_1 be the midpoint of $[a, b]$. That is,

$$x_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2} \quad \textbf{(first approximation)}.$$

If $f(x_1) = 0$, then $\alpha = x_1$, and we are done. If $f(x_1) \neq 0$, then $f(x_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$.

1. When $f(a_1)$ and $f(x_1)$ have the same sign, $\alpha \in (x_1, b_1)$, and we set $a_2 = x_1$ and $b_2 = b_1$.
2. When $f(a_1)$ and $f(x_1)$ have opposite signs, $\alpha \in (a_1, x_1)$, and we set $a_2 = a_1$ and $b_2 = x_1$.

We then reapply the process to the interval $[a_2, b_2]$ to get **second approximation** p_2.

How to apply bisection algorithm?

- An interval $[a, b]$ must be found with $f(a).f(b) < 0$.
- As at each step the length of the interval known to contain a zero of f is reduced by a factor of 2, it is advantageous to choose the initial interval $[a, b]$ as small as possible. For, example, if $f(x) = 2x^3 - x^2 + x - 1$, we have both

$$f(-4).f(4) < 0 \text{ and } f(0).f(1) < 0,$$

so the bisection algorithm could be used on either on the intervals $[-4, 4]$ or $[0, 1]$. However, starting the bisection algorithm on $[0, 1]$ instead of $[-4, 4]$ will reduce by 3 the number of iterations required to achieve a specified accuracy.

Bisection Method

Theorem 21.

Let f be a continuous function defined on $[a, b]$ and $f(a).f(b) < 0$. The bisection method generates a sequence $\{x_n\}_{n=1}^{\infty}$ approximating a zero α of f with

$$|x_n - \alpha| \le \frac{b-a}{2^n}, \text{ when } n \ge 1.$$

By Sandwich Theorem, the sequence $\{p_n\}$ of approximations converges to α with $f(\alpha) = 0$.

The method has the important property that **it always converges to a solution**, and for that reason it is often used as a starter for the more efficient methods.

But it is slow to converge.

Number of Iterations Needed?

To determine the number of iterations necessary to solve

$$f(x) = x^3 + 4x^2 - 10 = 0$$

with accuracy 10^{-3} using $a_1 = 1$ and $b_1 = 2$ requires finding an integer N that satisfies

$$|p_N - \alpha| \leq 2^{-N}(b - a) = 2^{-N} < 10^{-3}.$$

A simple calculation shows that ten iterations will ensure an approximation accurate to within 10^{-3}.

Again, it is important to keep in mind that the error analysis gives only a bound for the number of iterations, and in many cases this bound is much larger than the actual number required.

Continuous Function Theorem for Sequences

The following theorem states that applying a continuous function to a convergent sequence produces a convergent sequence.

Theorem 22 (Continuous Function Theorem for Sequences).

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$, that is,

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

Proof of "Continuous Function Theorem for Sequences"

Let $\varepsilon > 0$ be given.

Since f is continuous at L, there exists $\delta > 0$ such that

$$|x - L| < \delta \implies |f(x) - f(L)| < \varepsilon.$$

As $a_n \to L$, by applying the defintion for $\varepsilon = \delta$, there exists a positive integer N such that

$$|a_n - L| < \delta, \quad \text{for all } n > N.$$

Thus for all $n > N$, we have

$$|f(a_n) - f(L)| < \varepsilon.$$

This proves that $\{f(a_n)\}$ converges to $f(L)$.

How to use the Continuous Function Theorem for Sequences?

Let $\{b_n\}$ be a sequence.

If we are able to find a sequence $\{a_n\}$ converging to L and find a function f which is continuous at L and defined of all a_n, then the limit of $\{b_n\}$ is $f(L)$.

Example 23.

By "Continuous Function Theorem for Sequences", $\sqrt{\frac{n+1}{n}} \to 1$ *because* $\frac{n+1}{n} \to 1$ *and* $f(x) = \sqrt{x}$ *is continuous at* $x = 1$.

Example 24.

The sequence $\{1/n\}$ *converges to* 0. *By taking* $a_n = 1/n, f(x) = 2^x$, *and* $L = 0$ *in the "Continuous Function Theorem for Sequences", we see that* $2^{1/n} = f(1/n) \to f(L) = 2^0 = 1$. *The sequence* $\{2^{1/n}\}$ *converges to* 1.

In the above example, if 2 is replaced by any positive real 'a', we still get the same limit as 1:

Let a be a positive real number. By considering the continuous function $f(x) = a^x$, we can show that $\{a^{1/n}\}$ converges to 1.

An Application of Sequences - Sequential Criterion for Continuity

Theorem 25 (Sequential Criterion for Continuity).

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $x = c$ if and only if every sequence $\{a_n\}$ in D that converges to c, the sequence $\{f(a_n)\}$ converges to $f(c)$.

Exercise

Exercise 26.

Define $g : \mathbb{R} \to \mathbb{R}$ *by*

$$g(x) = \begin{cases} 2x & \textit{if } x \textit{ is rational} \\ x + 3 & \textit{if } x \textit{ is irrational.} \end{cases}$$

Find all points at which g *is continuous.*

To solve the exercise given above, we are using that fact that, in the set $\mathbb{R}$ of real numbers, the set $\mathbb{Q}$ of rational numbers and the set $\mathbb{R} \backslash \mathbb{Q}$ (the complement of $\mathbb{Q}$) of irrational numbers are dense.

Definition 27.

A subset D *of* $\mathbb{R}$ *is said to be* dense *if for every* $x \in \mathbb{R}$ *and every* $\varepsilon > 0$, *there exists* $y \in D$ *such that* $|x - y| < \varepsilon$.

Solution for Exercise 26

Note that for all $x \in \mathbb{R}$, $y = 2x$ and $y = x + 3$ are lines which intersect at $x = 3$. It seems reasonable to suspect that g is only continuous at $x = 3$.

g is continuous at $x = 3$:

Given $\varepsilon > 0$, we need to find $\delta > 0$ such that

$$|x - 3| < \delta \implies |g(x) - g(3)| < \varepsilon.$$

If x is rational, then $|g(x) - g(3)| = |2x - 6| = 2|x - 3|$. If x is irrational, then $|g(x) - g(3)| = |(x + 3) - 6| = |x - 3|$.

So, given $\varepsilon > 0$, set $\delta = \varepsilon/2$. Then for any real x (either rational or irrational) satisfying

$$|x - 3| < \delta \implies |g(x) - g(3)| < \varepsilon.$$

Hence g is continuous at $x = 3$.

Solution for Exercise 26 (contd...)

g is discontinuous at any $x = c \neq 3$:

We don't use "$\varepsilon - \delta$" definition here (which you have studied in the last semester). Instead, we use "Sequential Criterion for Continuity".

Since the set $\mathbb{Q}$ of rational numbers and the set $\mathbb{R}\backslash\mathbb{Q}$ (the complement of $\mathbb{Q}$) of irrational numbers are dense, there exist sequences $\{q_n\}$ in $\mathbb{Q}$ and s_n in $\mathbb{R}\backslash\mathbb{Q}$ such that $q_n \to c$ and $s_n \to c$.

However, $f(q_n) = 2q_n \to 2c$, whereas $f(s_n) = s_n + 3 \to c + 3$.

Since $c \neq 3$, we see that $2c \neq c + 3$. This cannot occur if f is continuous at $x = c$.

Finding Limits of Sequences Using Limits of Functions

In order to find the limit of $\{a_n\}$, sometimes we shall adopt the following technique:

We may consider a function f such that $f(n) = a_n$, for all $n \geq N$ (for some N).

If $\lim_{x\to\infty} f(x)$ exists, say L, (L can be any number including ∞ or $-\infty$), then we can say that the sequence $\{a_n\}$ converges to L, or $\{a_n\}$ diverges to ∞, or $-\infty$.

Why is this useful?

We can use L'Hôpital's Rule to find $\lim_{x\to\infty} f(x)$.

Recalling Indeterminate Forms and L'Hôpital's Rule

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or $+\infty$.

The rule is known today as **L'Hôpital's Rule**. He was French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ can be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate.

We use $0/0$ as a notation for an expression known as an **indeterminate form.** There are seven indeterminate forms which are typically considered in the literature:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^0,\ 1^\infty,\ \text{and } \infty^0.$$

Theorem 28 (L'Hôpital's Rule (First Form)).

Suppose that $f(a) = g(a) = 0$, *that* $f'(a)$ *and* $g'(a)$ *exist, and that* $g'(a) \neq 0$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Sometimes after differentiation, the new numerator and denominator both equal zero at $x = a$. In these cases, we apply a stronger form of L'Hôpital's Rule.

L'Hôpital's Rule

Theorem 29 (L'Hôpital's Rule (Stronger Form)).

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(a) \neq 0$ if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side exists.

L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by L'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$.

But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating.

L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Indeterminate Form ∞/∞

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an ambiguous expression like $\infty/\infty, \infty.0$, or $\infty - \infty$, instead of $0/0$.

L'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right side exists.

In the notation $x \to a$, a may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

Indeterminate Form $\infty.0$ and $\infty - \infty$

Sometimes these forms ($\infty.0$ or $\infty - \infty$) can be handled by using algebra by converting them to $0/0$ or ∞/∞ form.

Here again we do not mean to suggest that $\infty.0$ or $\infty - \infty$ is a number. They are only notations for functional behaviours when considering limits.

The next theorem enables us to use L'Hôpital's Rule to find the limits of some sequences. It formalizes the connection between $\lim_{n\to\infty} a_n$ and $\lim_{x\to\infty} f(x)$.

Theorem 30 (Using L'Hôpital's Rule).

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

Proof

Suppose that $\lim_{x\to\infty} f(x) = L$.

Then for each positive number ε there is a number M such that for all x.

$$x > M \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Let N be an integer greater than M and greater than or equal to n_0.

Then

$$n > N \quad \Rightarrow \quad a_n = f(n)$$

and

$$|a_n - L| \quad = \quad |f(n) - L| < \varepsilon.$$

Example 31.

Show that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

The function $\frac{\ln x}{x}$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by L'Hôpital's Rule for sequences, $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n}$ will equal $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$ if the latter exists.

A single application of L'Hôpital's Rule shows that

$$\lim_{n\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

L'Hôpital's Rule

When we use L'Hôpital's Rule to find the limit of a sequence, **we often treat n as a continuous real variable and differentiate directly with respect to n.** This saves us from having to rewrite the formula for a_n as we have done it in the example above.

Example 32.

Find

$$\lim_{n\to\infty} \frac{2^n}{5n}.$$

By L'Hôpital's Rule (differentiating with respect to n),

$$\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n.\ln 2}{5} = \infty.$$

Example 33.

Find limit of each of the following sequences, if it exists.

(a) $\lim\limits_{n\to\infty} \sqrt[n]{n}$.

(b) $\lim\limits_{n\to\infty} x^{1/n}$, *for a fixed positive real* x.

(a) $$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \exp\left[\frac{1}{n}\ln n\right] = \exp\left[\lim_{n\to\infty} \frac{1}{n}\ln n\right] = \exp(0) = 1.$$

(b) Let $x > 0$ be fixed.

$$\lim_{n\to\infty} x^{1/n} = \lim_{n\to\infty} \exp\left[\frac{1}{n}\ln x\right] = \exp\left[\lim_{n\to\infty} \frac{1}{n}\ln x\right] = \exp(0) = 1.$$

Example 34.

Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

The limit leads to the indeterminate form 1^∞. We can apply L'Hôpital's Rule if we first change the form to $\infty \, . \, 0$ by taking the natural logarithms of a_n

$$\begin{aligned} \ln a_n &= \ln \left(\frac{n+1}{n-1}\right)^n \\ &= n \ln \left(\frac{n+1}{n-1}\right). \end{aligned}$$

L'Hôpital's Rule

Then,

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \quad \left[\infty.0 \ \ \text{form}\right] \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \quad \left[\frac{0}{0} \ \ \text{form}\right] \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \quad \left[\text{By L'Hôpital's Rule}\right] \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.
\end{aligned}$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, by "Continuous Function Theorem for Sequences", we get that $a_n = e^{\ln a_n} \to e^2$. The sequence $\{a_n\}$ converges to e^2.

Factorial Notation

The notation $n!$ ("n factorial") means the product $1.2.3. \ldots .n$ of the integers from 1 to n. Notice that $(n+1)! = (n+1).n!$. Thus, $4! = 1.2.3.4 = 24$ and $5! = 1.2.3.4.5 = 5.4! = 120$.

We define $0!$ to be 1. **Factorials grow even faster than exponential**, as the table suggests.

n	e^n (rounded)	$n!$
1	3	1
5	148	120
10	22,026	3,628,800
20	$4.9x10^8$	$2.4x10^{18}$

Using $\varepsilon - N$ definition, prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Let $\sqrt[n]{n} = 1 + h_n$ where $h_n \geq 0$. Then for all n, (since $h_n \geq 0$), we have

$$n = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + \cdots + h_n^n > \frac{n(n-1)}{2}h_n^2.$$

Hence $h_n^2 < \frac{2}{n-1}$ for all $n \geq 2$, so $|h_n| < \sqrt{\frac{2}{n-1}}$, for all $n \geq 2$.

Let $\varepsilon > 0$ be given. Then

$$|h_n| < \sqrt{\frac{2}{n-1}} < \varepsilon, \quad \text{when } n > 1 + 2/\varepsilon^2.$$

For any positive integer N greater than $1 + 2/\varepsilon^2$, we have

$$|\sqrt[n]{n} - 1| = |h_n| < \varepsilon, \quad \text{for all } n > N.$$

Hence $\sqrt[n]{n} \to 1$.

Using $\varepsilon - N$ definition, prove that $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$.

<u>**Case :** $0 < x < 1$</u>

Let $y = 1/x$. Since $x < 1, y > 1$. Let $\sqrt[n]{y} = 1 + h_n$, when $h_n > 0$.

$$\begin{aligned} y &= (1+h_n)^n = 1 + nh_n + \frac{n(n+1)}{2}h_n^2 + \cdots + h_n^n \\ &> 1 + nh_n \quad (\text{since} \quad h_n > 0), \qquad \text{for all } n. \end{aligned}$$

So $h_n < \frac{y-1}{n}$, for all n.

Let $\varepsilon > 0$ be given. Then $|h_n| < \frac{y-1}{n} < \varepsilon$, when $n > \frac{y-1}{\varepsilon}$.

For any positive integer N greater than, $\frac{y-1}{\varepsilon} = \frac{1/x-1}{\varepsilon}$, we have

$$|\sqrt[n]{y} - 1| = |h_n| < \varepsilon, \quad \text{for all } n > N.$$

Using $\varepsilon - N$ definition, prove that $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$.

Case : $x > 1$

By the argument given in the above case, for any given $\varepsilon > 0$, there is an integer N greater than $\frac{x-1}{\varepsilon}$, we have

$$|\sqrt[n]{x} - 1| < \varepsilon, \qquad \text{for all } n > N.$$

Case : $x = 1$

The case is trivial, since $\{x^{1/n}\}$ is the 1-constant sequence, converging to 1.

For given $\varepsilon > 0$, any positive integer N satisfying

$$n > N \Rightarrow |1 - 1| = 0 < \varepsilon.$$

Using $\varepsilon - N$ definition, prove that $\lim_{n\to\infty} x^n$ converges only when $-1 < x \le 1$.

Exercise 35.

Prove that $\lim_{n\to\infty} x^n$ *converges to* 0 *when* $|x| < 1$.

Proof. <u>**Case :** $|x| < 1$</u>

Let $|x| = \frac{1}{1+h}$ where $h > 0$. Therefore $|x|^n = \frac{1}{(1+h)^n} \le \frac{1}{1+nh}$, for all n.

Let $\varepsilon > 0$ be given. Then $\frac{1}{1+nh} < \varepsilon$ when $n > \frac{(1/\varepsilon - 1)}{h}$.

For any positive integer N greater than $\frac{(1/\varepsilon - 1)}{h}$, we have

$$|x^n - 0| = \frac{1}{1+nh} < \varepsilon, \text{ for all } n > N.$$

Hence, $\{x^n\}$ converges to 0.

Using $\varepsilon - N$ definition, prove that $\lim_{n \to \infty} x^n$ converges only when $-1 < x \leq 1$.

Case : $x = 1$

When $x = 1$, evidently $x^n = 1$. Therefore the sequence converges to 1.

Case : $x > 1$

Let $x = 1 + h$, $h > 0$. Then $x^n = (1 + h)^n > 1 + nh$, for all n.

Let M be a positive number (however large) such that $1 + nh > G$.

For any $G > 0$, there is a positive integer N such that $x^n > G$, for all $n > N$. Hence the sequence diverges to ∞.

Case : $x = -1$

When $x = -1$, the equation $\{(-1)^n\}$ oscillates finitely.

Using $\varepsilon - N$ definition, prove that $\lim_{n\to\infty} x^n$ converges only when $-1 < x \leq 1$.

<u>**Case :** $x < -1$</u>

Let $x = -y$ so that $y > 1$.

Thus we get the sequence $\{(-1)^n y^n\}$ which have both positive and negative terms.

The sequence is unbounded and the numerical values of the terms can be made greater than any number (however large). Thus, it oscillates infinitely.

Hence the sequence $\{x^n\}$ converges only when $-1 < x \leq 1$.

Using L'Hôpital's Rule, prove that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ (any x).

$$
\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \exp\left(\lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n\right)\\
&= \exp\left(\lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right)\right)\\
&= \exp\lim_{n\to\infty}\left(\frac{\ln(1+\frac{x}{n})}{\frac{1}{n}}\right) \qquad \left[\frac{\infty}{\infty} form\right]\\
&= \exp\lim_{n\to\infty}\frac{\frac{1}{1+\frac{x}{n}\cdot}\cdot\frac{-x}{n^2}}{\frac{-1}{n^2}}\\
&= \exp\lim_{n\to\infty}\frac{x}{1+\frac{x}{n}}\\
&= e^x
\end{aligned}
$$

Using $\varepsilon - N$ definition, prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x).

Let $\varepsilon > 0$ be given. First we choose a positive integer N such that

$$N \geq 2|x|.$$

Let $K = \frac{|x|^N}{N!}$. Note that $\frac{|x|}{K} \leq \frac{1}{2}$, for all $K \geq N$.

Now we can choose a positive integer $N_1 > N$ such that

$$\left(\frac{1}{2}\right)^{N_1 - N} K < \varepsilon.$$

Then for all $n \geq N_1$, we have

$$\frac{|x|^n}{n!} = \frac{|x|^{n-N}}{n(n-1)\cdots(n-N+1)} \quad \frac{|x|^N}{N!} \leq \left(\frac{1}{2}\right)^{n-N} K < \varepsilon.$$

Recursive Definitions

So far, we have calculated each a_n directly from the value of n.

But sequences are often defined **recursively** by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a **recursion formula**, for calculating any later term from terms that precede it.

13th-century Italian mathematician Fibonacci posed the following problem, concerning the breeding of rabbits :

Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months.

If we start with one newborn pair, how many pairs of rabbits will we have in the *n*th month ? The answer can be given in a recursion formula.

The statements $a_1 = 1, a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ define the sequence $1, 1, 2, 3, 5, \ldots$ of **Fibonacci numbers.** With $a_1 = 1$ and $a_2 = 1$, we have $a_3 = 1 + 1 = 2, a_4 = 2 + 1 = 3, a_5 = 3 + 2 = 5$, and so on.

Sequence Constructed Recursively

Example 36.

(a) *The statements* $a_1 = 1$ *and* $a_n = a_{n-1} + 1$ *define the sequence* $1, 2, 3, \ldots, n, \ldots$ *of positive integers. With* $a_1 = 1$, *we have* $a_2 = a_1 + 1 = 2, a_3 = a_2 + 1 = 3,$ *and so on.*

(b) *The statements* $a_1 = 1$ *and* $a_n = n \cdot a_{n-1}$ *define the sequence* $1, 2, 6, 24, \ldots, n!, \ldots$ *of factorials. With* $a_1 = 1$, *we have* $a_2 = 2 \cdot a_1 = 2, a_3 = 3 \cdot a_2 = 6, a_4 = 4 \cdot a_3 = 24,$ *and so on.*

Bounded Nondecreasing Sequence

The terms of a general sequence can bounce around, sometimes getting larger, sometimes smaller. An important special kind of sequence is one for which each term is at least as large as its predecessor.

Definition 37 (Nondecreasing Sequence).

A sequence $\{a_n\}$ *with the property that* $a_n \leq a_{n+1}$ *for all* n *is called a* nondecreasing sequence.

Example 38 (Nondecreasing Sequence).

(a) *The sequence* $1, 2, 3, \ldots, n, \ldots$ *of natural numbers.*

(b) *The sequence* $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$.

(c) *The constant sequence* $\{3\}$.

There are two kinds of nondecreasing sequences – those whose terms increase beyond any finite bound and those whose terms do not.

Bounded, Upper Bound, Least Upper Bound

Definition 39.

A sequence $\{a_n\}$ *is* bounded from above *if there exists a number* M *such that* $a_n \leq M$ *for all* n. *The number* M *is an* upper bound *for* $\{a_n\}$. *If* M *is an upper bound for* $\{a_n\}$ *but no number less than* M *is an upper bound for* $\{a_n\}$, *then* M *is the* least upper bound *for* $\{a_n\}$.

Example 40 (Applying the Definition for Boundedeness).

(a) *The sequence* $1, 2, 3, \ldots, n, \ldots$ *has no upper bound.*

(b) *The sequence* $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ *is bounded from above by* $M = 1$. *No number less than* 1 *is an upper bound for the sequence, so* 1 *is the least upper bound, from the following exercise.*

Exercise 41.

Show that if M is a number less than 1, then the terms of $\{n/(n+1)\}$ eventually exceed M.

Proof

Let $0 < M < 1$ and N be an integer greater than $\frac{M}{1-M}$.

Then

$$n > N \implies n > \frac{M}{1-M} \implies \frac{n}{n+1} > M.$$

Since $\frac{n}{n+1} < 1$ for every n, this proves that 1 is the least upper bound for the sequence $\left\{\frac{n}{n+1}\right\}$.

Bounded Nondecreasing Sequence

A nondecreasing sequence that is bounded from above always has the least upper bound. This is the **completeness property of the real numbers**, We will prove that if L is the least upper bound then the sequence converges to L.

Theorem 42 (Nondecreasing Sequence Theorem).

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

The above theorem implies that a nondecreasing sequence converges when it is bounded from above. It diverges to infinity if it is not bounded from above.

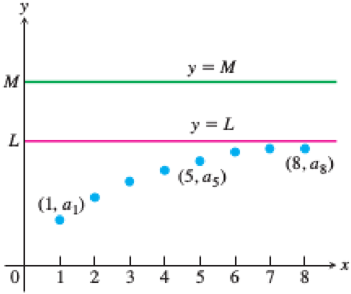
Monotonic Sequence Theorem

Theorem 43 (Monotonic Sequence Theorem).

If a sequence $\{a_n\}$ *is both bounded and monotonic, then the sequence converges.*

Proof of Nondecreasing Sequence Theorem / Monotonic Sequence Theorem

Suppose we plot the points $(1, a_1), (2, a_2), \ldots, (n, a_n), \ldots$ in the xy-plane. If M is an upper bound of the sequence, all these points will lie on below the line $y = M$.



If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \leq M$.

Proof of Nondecreasing Sequence Theorem / Monotonic Sequence Theorem (contd...)

The line $y = L$ is the lowest such line. None of the points (n, a_n) lies above $y = L$, but some do lie above any lower line $y = L - \varepsilon$, if ε is a positive number. The sequence converges to L because

(a) $a_n \leq L$ for all values of n and

(b) Given any $\varepsilon > 0$, there exists at least one integer N for which $a_N > L - \varepsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that $a_n \geq a_n > L - \varepsilon$, for all $n \geq N$. Thus, all the numbers a_n beyond the Nth number lie within the ε-band of L. This is precisely the condition for L be the limit of the sequence $\{a_n\}$.

Bounded, Lower Bound, Greatest Lower Bound

Definition 44.

A sequence of numbers $\{a_n\}$ *in which* $a_n \geq a_{n+1}$ *for every* n *is called a* nonincreasing sequence. *A sequence* $\{a_n\}$ *is* bounded from below *if there is number* M *with* $M \geq a_n$ *for every* n. *Such a number* M *is called a* lower bound *for the sequence.*
If m *is a lower bound for* $\{a_n\}$ *but no number greater than* m *is a lower bound for* $\{a_n\}$*, then* m *is the* greatest lower bound *for* $\{a_n\}$.

Theorem 45 (Nonincreasing Sequence Theorem).

A nonincreasing sequence of real numbers converges if and only if it is bounded from below. If a nonincreasing sequence converges, it converges to its greatest lower bound.

Proof is similar to the proof of Nondecreasing Sequence Theorem.

When to apply Nondecreasing Sequence Theorem / Monotonic Sequence Theorem ?

To apply Nondecreasing Sequence Theorem / Monotonic Sequence Theorem to the sequence $\{a_n\}$, we write first few terms of the sequence $\{a_n\}$ and observe the pattern.

Then using mathematical induction, we may prove that it is increasing / decreasing and bounded from above / below.

Observations

1. The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.
2. Every monotonic sequence either converges or diverges and is never oscillatory.
3. Every monotonically increasing sequence which is not bounded above diverges to $+\infty$.
4. Every monotonically decreasing sequence which is not bounded below diverges to $-\infty$.

Exercise 46.

Show that the sequence $\{a_n\}$ *defined by the recursion formula*

$$a_{n+1} = \sqrt{3a_n}, \quad a_1 = 1$$

converges to 3*.*

Solution

Clearly, $\sqrt{3} = a_2 \geq a_1 = 1$.

Suppose $a_{m+1} \geq a_m \implies \sqrt{3a_{m+1}} \geq \sqrt{3a_m} \implies a_{m+2} \geq a_{m+1}$.

Thus by mathematical induction, $a_{n+1} \geq a_n$, for all n.

Clearly, $a_1 < 3$, $a_2 < 3$, $a_3 < \sqrt{3\sqrt{3}} < 3$. **Use can use powers of 3.**

Again, by mathematical induction,

$$0 < a_n < 3, \quad \text{for all } n.$$

Hence by nondecreasing sequence theorem, $\{a_n\}$ converges.

Solution (contd...)

Suppose that $\{a_n\}$ converges to L. Then $\{a_{n+1}\}$ also converges to L.

Since $a_{n+1} = \sqrt{3a_n}$, as $n \to \infty$ we get $L = \sqrt{3L}$.

So, $L = 0$, or, $L = 3$.

Since $a_n \geq 1$, for all n, 0 cannot be a limit.

Thus $\{a_n\} \to 3$, as $n \to \infty$.

Example 47.

Some examples of convergent sequences are given below.

1. $\frac{\ln(n^2)}{n} = \frac{2\ln n}{n} \to 2.0 = 0$
2. $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$
3. $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1.1 = 1$
4. $\left(-\frac{1}{2}\right)^n \to 0$
5. $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$
6. $\frac{100^n}{n!} \to 0$

Exercise 48 (Finding Limits).

Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = 2 + (0.1)^n$
2. $a_n = \frac{2n+1}{1-3\sqrt{n}}$
3. $a_n = \frac{1-5n^4}{n^4+8n^3}$
4. $a_n = \frac{1-n^3}{70-4n^2}$
5. $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$

Solution

1. $a_n = 2 + (0.1)^n \to 2$, hence it converges.
2. $a_n = \frac{2n+1}{1-3\sqrt{n}} \to -\infty$, hence it diverges.
3. $a_n = \frac{1-5n^4}{n^4+8n^3} \to -5$, hence it converges.
4. $a_n = \frac{1-n^3}{70-4n^2} \to \infty$, hence it diverges.
5. $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right) \to 6$, hence it converges.

Exercise 49 (Finding Limits).

Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = \sqrt{\frac{2n}{n+1}}$
2. $a_n = \frac{\sin^2 n}{2^n}$
3. $a_n = \sqrt[n]{10n}$
4. $a_n = \ln n - \ln(n+1)$
5. $a_n = \frac{n!}{n^n}$ *(Hint: Compare with* $1/n$*.)*

Solution

1. $\lim a_n = \sqrt{\lim \frac{2n}{n+1}} = \sqrt{2}$.
2. $0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$, by Sandwich theorem, $\{a_n\}$ converges to 0.
3. $\lim a_n = \lim \sqrt[n]{10}\sqrt[n]{n} \to 1.1 = 1$, so $\{a_n\}$ converges to 1.
4. $\lim a_n = \lim \ln\left(\frac{n}{n+1}\right) \to \ln 1 = 0$, so $\{a_n\}$ converges to 0.
5. $0 \leq \lim \frac{1.2.3\cdots n}{n.n.n\cdots n} \leq \lim \frac{1}{n}$, by Sandwich theorem, $\{a_n\}$ converges to 0.

Exercise 50.

Which of the sequences $\{a_n\}$ converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = \frac{n!}{2^n.3^n}$
2. $a_n = \left(\frac{1}{n}\right)^{1/(\ln n)}$
3. $a_n = \left(\frac{x^n}{2n+1}\right)^{1/n}, x > 0$
4. $a_n = \tanh n$
5. $a_n = \frac{1}{\sqrt{n}} \tan^{-1} n$
6. $a_n = \frac{(\ln n)^{200}}{n}$
7. $a_n = \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$
8. $a_n = \frac{1}{n} \int_1^n \frac{1}{x}\ dx$
9. $a_n = \int_1^n \frac{1}{x^p}\ dx,\ p > 1$

Solution

1. $\lim \frac{n!}{2^n.3^n} = \lim \frac{1}{6^n/n!} = \infty$, diverges.
2. $\lim \left(\frac{1}{n}\right)^{1/(\ln n)} = \lim \exp\left(\frac{1}{\ln n}\ln(\frac{1}{n})\right) = \lim \exp\left(\frac{\ln 1 - \ln n}{\ln n}\right) = e^{-1}$, converges.
3. $\lim \left(\frac{x^n}{2n+1}\right)^{1/n} = \lim x\left(\frac{1}{2n+1}\right)^{1/n} = x\ \exp\lim\left(\frac{-2}{(2n+1)^2}\right) = xe^0 = x, x > 0$, converges.
4. $\lim \tanh n = \lim \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim \frac{e^{2n}-1}{e^{2n}+1} = 1$, converges.
5. $\lim \frac{1}{\sqrt{n}} \tan^{-1} n = 0.\frac{\pi}{2} = 0$, converges.
6. $\lim \frac{(\ln n)^{200}}{n} = \lim \frac{200(\ln n)^{199}}{n} = \cdots = \lim \frac{200!}{n} = 0$, converges.
7. $\lim \frac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}} = \lim \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{-1-n} = -2$, converges.
8. $\lim \frac{1}{n}\int_1^n \frac{1}{x}\ dx = \lim \frac{\ln n}{n} = 0$, converges.
9. $\lim \int_1^n \frac{1}{x^p}\ dx = \lim \left[\frac{1}{p-1}\frac{1}{x^{p-1}}\right]_1^n = \lim \frac{1}{p-1}\left[\frac{1}{n^{p-1}} - 1\right] = \frac{1}{p-1}, p > 1$, converges.

Exercise 51.

Assume that each of the following sequences is recursively defined. Find its limit.

1. $a_1 = 1, a_{n+1} = \frac{a_n+6}{a_n+2}$
2. $a_{n+1} = \sqrt{8 + 2a_n}$. *Discuss the limit when* $a_1 = -4$ *and* $a_1 = 0$.
3. $2, 2 + \frac{1}{2}, 2 + \frac{1}{2+\frac{1}{2}}, 2 + \frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \cdots$
4. $\sqrt{3}, \sqrt{3+\sqrt{3}}, \sqrt{3+\sqrt{3+\sqrt{3}}}, \sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}, \cdots$

Solution

1. Since $\{a_n\}$ converges, the limit is 2.
2. Since $\{a_n\}$ converges, the limit is 4 for $a_1 = -4$ and the limit is 0 for $a_1 = 0$.
3. By induction, we have $a_n \geq 2$ for all $n \geq 1$. Since $\{a_n\}$ is Cauchy, the sequence $\{a_n\}$ converges, the limit is $1 + \sqrt{2}$.
   Some observations : By looking at the terms of the sequence, we have the following observations :
   (a) odd terms form an increasing sequence
   (b) even terms form a decreasing sequence
   (c) all odd terms less than all even terms

   So both odd term subsequence and even term subsequence converge to the same limit.
4. By induction, $a_n \leq 3$ for all $n \geq 1$ and $a_n \leq a_{n+1}$ for all $n \geq 1$, hence $\{a_n\}$ converges. The limit is $\frac{1+\sqrt{13}}{2}$.

Exercise 52.

The first term of a sequence is $x_1 = 1$. *Each succeeding term is the sum of all those that come before it:*

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for x_n *that holds for* $n \geq 2$.

Solution

The terms are

$$1, 1, 2, 4, 8, 16, 32, \ldots = 1, 2^0, 2^1, 2^2, 2^3, \ldots .$$

$\implies x_1 = 1$ and $x_n = 2^{n-2}$, for $n \geq 2$.

Exercise 53.

A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a+2b}{a+b}, \ldots.$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be, respectively, the numerator and the denominator of the nth fraction $r_n = x_n/y_n$.

(a) *Verify that $x_1^2 - 2y_1^2 = -1, x_2^2 - 2y_2^2 = +1$ and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then $(a+2b)^2 - 2(a+b)^2 = +1$ or -1, respectively.*

(b) *The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit? (Hint: Use part (a) to show that $r_n^2 - 2 = (1/y_n)^2$ and that y_n is not less than n.)*

Solution

Let $f(a, b) = (a + 2b)^2 - 2(a + b)^2 = 2b^2 - a^2$.

(a) $x_1^2 - 2y_1^2 = -1$ and $x_2^2 - 2y_2^2 = +1$ are verified for $r_1 = \frac{x_1}{y_1} = \frac{1}{1}$ and $r_2 = \frac{x_2}{y_2} = \frac{3}{2}$.
If $a^2 - 2b^2 = -1$ or $+1$, then $f(a, b) = +1$ or -1, respectively.

(b) $r_n^2 - 2 = \left(\frac{a+2b}{a+b}\right)^2 - 2 = -\frac{(a^2-2b^2)}{(a+b)^2}$. Note that $\frac{a}{b}$ is the predecessor of $\frac{a+2b}{a+b}$.
If $a^2 - 2b^2 = -1$ or $+1$, then $(a + 2b)^2 - 2(a + b)^2$ is 1 or -1. Hence

$$r_n^2 - 2 = -\frac{(a^2 - 2b^2)}{(a + b)^2} = \frac{\pm 1}{y_n}$$

which implies that $r_n = \sqrt{2 \pm (\frac{1}{y_n})^2}$.
Verifty that $y_n \geq n$ for all n, hence $\frac{1}{y_n} \to 0$ as $n \to \infty$. Thus $r_n \to \sqrt{2}$.

Sequences Generated by Newton's Method

Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and constructs from it a sequence of numbers $\{x_n\}$ that under favorable circumstances converges to a zero of f. The recursion formula for the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For $a > 0$, one can easily show that the recursion formula for $f(x) = x^2 - a$ can be written as

$$x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2}.$$

Exercise 54.

The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Does the sequence converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

1. $x_0 = 1, x_{n+1} = x_n - [(\sin x_n - x_n^2)/(\cos x_n - 2x_n)]$
2. $x_0 = 1, x_{n+1} = x_n - \frac{x_n^2-2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$
3. $x_0 = 1, x_{n+1} = x_n - \frac{\tan x_n - 1}{\sec^2 x_n}$
4. $x_0 = 1, x_{n+1} = x_n - 1.$

Solution

1. The sequence converges to a solution of the equation $\sin x - x^2 = 0$.
2. $f(x) = x^2 - 2$, the sequence converges to $\sqrt{2}$.
3. $f(x) = \tan x - 1$, the sequence converges to $0.7853981 \approx \pi/4$.
4. $f(x) = e^x$, the sequence diverges.

Exercise 55.

1. *Suppose that $f(x)$ is differentiable for all x in $[0,1]$ and that $f(0)=0$. Define the sequence $\{a_n\}$ by the rule $a_n = nf(1/n)$. Show that*

$$\lim_{n\to\infty} a_n = f'(0).$$

Use the above result to find the limits of the following sequences:

(a) $a_n = n\tan^{-1}\frac{1}{n}$
(b) $a_n = n(e^{1/n} - 1)$
(c) $a_n = n\ln(1+\frac{2}{n})$

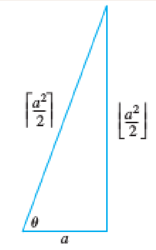
Solution

1. $\lim_{n\to\infty} nf(1/n) = \lim_{\Delta x\to 0} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = f'(0)$, where $\Delta x = \frac{1}{n}$.
   (a) $\lim n \tan^{-1} \frac{1}{n} = f'(0) = 1$, where $f(x) = \tan^{-1} x$.
   (b) $\lim n(e^{1/n} - 1) = f'(0) = e^0 - 1$, where $f(x) = e^x - 1$.
   (c) $\lim n \ln(1 + \frac{2}{n}) = f'(0) = 2$, where $f(x) = \ln(1 + 2x)$.

Exercise 56.

A triple of positive integers a, b, and c is called a Pythagorean triple *if* $a^2 + b^2 = c^2$. *Let a be an odd positive integer and let* $b = \lfloor \frac{a^2}{2} \rfloor$ *and* $c = \lceil \frac{a^2}{2} \rceil$ *be, respectively, the integer floor and ceiling for* $a^2/2$.

1. *Show that* $a^2 + b^2 = c^2$. *(Hint: Let* $a = 2n + 1$ *and express b and c in terms of n.)*
2. *By direct calculation, or by appealing to the figure here, find*
$$\lim_{a \to \infty} \frac{\lfloor \frac{a^2}{2} \rfloor}{\lceil \frac{a^2}{2} \rceil}.$$



Solution

1. If $a = 2n + 1$, then $b = \lfloor \frac{a^2}{2} \rfloor = 2n^2 + 2n$, $c = \lceil \frac{a^2}{2} \rceil = 2n^2 + 2n + 1$, hence $a^2 + b^2 = c^2$. (Verify!)
2. $$\lim_{a \to \infty} \frac{\lfloor \frac{a^2}{2} \rfloor}{\lceil \frac{a^2}{2} \rceil} = \lim_{n \to \infty} \frac{2n^2 + 2n}{2n^2 + 2n + 1} = 1.$$

Good Approximation for nth root of $n!$

Exercise 57.

Scottish mathematician James Stirling (1692-1770) showed that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}. \tag{2}$$

Show that

$$\lim_{n\to\infty} (2n\pi)^{1/(2n)} = 1$$

and hence, using Stirling's approximation (2), that

$$\sqrt[n]{n!} \approx \frac{n}{e}$$

for large values of n.

Solution

$$\lim_{n\to\infty}(2n\pi)^{1/(2n)} = \lim_{n\to\infty}\exp\left(\frac{\ln 2n\pi}{2n}\right) \to e^0 = 1.$$

From (2), we get that

$$\sqrt[n]{n!} \approx \frac{n}{e}(2n\pi)^{1/2n} \approx \frac{n}{e},$$

for large value n.

Exercise 58.

1. *Assuming that* $\lim_{n\to\infty}(1/n^c) = 0$ *if* c *is any positive constant, show that*

$$\lim_{n\to\infty} \frac{\ln n}{n^c} = 0$$

   if c *is any positive constant.*

2. *Prove that* $\lim_{n\to\infty}(1/n^c) = 0$ *if* c *is any positive constant. (Hint : if* $\varepsilon = 0.001$ *and* $c = 0.04$, *how large should* N *be to ensure that* $|1/n^c - 0| < \varepsilon$ *if* $n > N$*?)*

Solution

1. $\lim_{n\to\infty} \frac{\ln n}{n^c} = \lim_{n\to\infty} \frac{1/n}{cn^{c-1}} = \lim_{n\to\infty} \frac{1}{cn^c} = 0.$
2. For any $\varepsilon > 0$, there exists a positive integer $N = e^{-(\ln \varepsilon)/c}$ such that

$$n > e^{-(\ln \varepsilon)/c}.$$

Hence $\ln n > -\frac{\ln \varepsilon}{c} \implies \frac{1}{n^c} < \varepsilon$. Thus $\lim_{n\to\infty} (1/n^c) = 0$.

The zipper theorem

Exercise 59.

Prove the "zipper theorem" for sequences *: If* $\{a_n\}$ *and* $\{b_n\}$ *both converge to* L*, then the sequence*

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$$

converges to L*.*

Solution

Define $c_{2n} = b_n$ and $c_{2n-1} = a_n$, for $n = 1, 2, \ldots$.

Since $a_n \to L$ and $b_n \to L$, there are positive integers N_1 and N_2 respectively such that

$$|a_n - L| < \varepsilon, \quad \text{for all } n > N_1$$

and

$$|b_n - L| < \varepsilon, \quad \text{for all } n > N_2.$$

If $n > 1 + 2\max\{N_1, N_1\}$, then

$$|c_n - L| < \varepsilon, \quad \text{for all } n > N.$$

Hence $\{c_n\}$ converges to L.

Exercise 60.

Which of the following sequences converge and which diverge?

1. $a_n = 2 - \frac{2}{n} - \frac{1}{2^n}$
2. $a_n = ((-1)^n + 1)\left(\frac{n+1}{n}\right)$

Solution

1. Let's assume $a_n \leq a_{n+1}$ for all n. So $2 - \frac{2}{n} - \frac{1}{2^n} \leq 2 - \frac{2}{n+1} - \frac{1}{2^{n+1}}$ which gives that for all n

$$\frac{2}{n(n+1)} \geq -\frac{1}{2^{n+1}}.$$

Hence $\{a_n\}$ is non-decreasing. Also 2 is an upper bound. Thus $\{a_n\}$ converges to the lub.

2.

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2(1 + \frac{1}{n}) & \text{if } n \text{ is even} \end{cases}$$

so $\{a_n\}$ diverges.

Exercise 61.

The first term of a sequence is $x_1 = \cos(1)$. *The next terms are* $x_2 = x_1$ *or* $\cos(2)$, *whichever is larger, and*

$$x_3 = x_2 \quad \text{or} \quad \cos(3),$$

whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n+1)\}.$$

Solution

Verify that

$$x_n = \max\{\cos 1, \cos 2, \ldots, \cos n\}.$$

By the definition of maximum,

$$x_{n+1} \geq x_n$$

for all n.

Hence $\{x_n\}$ is non-decreasing and bounded from above by 1, so it converges.

Exercise 62.

Which of the sequences converge and which diverge?

1. $a_n = \dfrac{1 + \sqrt{2n}}{\sqrt{n}}$
2. $a_n = \dfrac{n+1}{n}$
3. $a_n = \dfrac{4^{n+1} + 3^n}{4^n}$
4. $a_1 = 1, a_{n+1} = 2a_n - 3$

Solution

1. Verify that $\{a_n\}$ is nonincreasing and bounded from below by $\sqrt{2}$. So it converges to glb.
2. Verify that $\{a_n\}$ is nonincreasing and bounded from below by 1. So it converges to glb.
3. Verify that $\{a_n\}$ is nonincreasing and bounded from below by 4, hence it converges to glb.
4. Verify that $a_n = -2^n + 3$ and $\{a_n\}$ is nonincreasing but not bounded from below, it diverges.

Uniqueness of least upper bounds

Exercise 63.

1. *Show that if* M_1 *and* M_2 *are least upper bounds for the sequence* $\{a_n\}$*, then* $M_1 = M_2$*. That is, a sequence cannot have two different least upper bounds.*
2. *Is it true that a sequence* $\{a_n\}$ *of positive numbers must converge if it is bounded from above? Give reasons for your answer.*

Solution

1. Use the fact that the least upper bound is also an upper bound.
2. No. The sequence $\{1, 2, 3, 1, 2, 3, \ldots\}$ is bounded from above but it does not converge. However, any increasing sequence which is bounded from above will converge.

Definition 64.

A sequence $\{a_n\}$ *is said to be a* Cauchy sequence *if to every positive number* ε *there corresponds an integer* N *such that for all* m *and* n,

$$m > N \text{ and } n > N \implies |a_m - a_n| < \varepsilon.$$

Equivalently, $\{a_n\}$ *is said to be a Cauchy sequence if given* $\varepsilon > 0$, *there exists a positive integer* m *such that* $|a_{m+p} - a_m| < \varepsilon$, *for all* $p \geq 0, p \in \mathbb{N}$.

Exercise 65.

Prove that every convergent sequence is a Cauchy sequence.

That is, prove that if $\{a_n\}$ *is a convergent sequence, then to every positive number* ε *there corresponds an integer* N *such that for all*

$$m > N \text{ and } n > N \implies |a_m - a_n| < \varepsilon.$$

Solution

Let $\{a_n\}$ be a convergent sequence, converging to some L.

Let $\varepsilon > 0$ be given.

As $a_n \to L$, there is a positive integer N such that

$$|a_n - L| < \varepsilon/2 \quad \text{for all } n > N.$$

Hence for all $n, m > N$, we have

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \varepsilon/2 + \varepsilon/2.$$

Thus $\{a_n\}$ is a Cauchy sequence.

We proved that every convergent sequence is a Cauchy sequence. **What about the converse ?** The converse is also true, which is shown in the following result (without proof).

Theorem 66 (Cauchy's general principle of convergence).

A necessary and sufficient condition for the convergence of a sequence $\{a_n\}$ is that, for each $\varepsilon > 0$, there exists a positive integer N such that

$$|a_m - a_n| < \varepsilon, \quad \text{for all } n, m > N.$$

How is "Cauchy's general principle of convergence" useful?

We can apply definition to test convergence of a sequence to a given limit L. But **in case, the limit L is not known, nor can any guess be made of the same, the above theorem can be used.** It involves only the terms of the sequence and is useful for determining whether a sequence converges or not.

Exercise 67.

Show that the sequence $\{a_n\}$, *where*

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

cannot converge.

Solution

Suppose that the sequence $\{a_n\}$ converges. For $\varepsilon = \frac{1}{2}$, there is a positive integer N such that

$$|a_m - a_n| < \frac{1}{2}, \quad \text{for all } n, m > N.$$

But

$$|a_m - a_{2m}| = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} > \frac{1}{2m}m = \frac{1}{2}$$

which contradicts that "every convergent sequence is a Cauchy sequence".

Hence the sequence cannot converge.

Exercise 68.

Show that the sequence $\{a_n\}$, *where*

$$a_n = \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}$$

converges to 0.

Solution

$$\frac{n}{(2n)^2} \leq a_n \leq \frac{n}{n^2} \quad \text{for all } n.$$

Hence $\frac{1}{4n} \leq a_n \leq \frac{1}{n}$ for all n.

By Sandwich theorem, $\{a_n\}$ converges to 0.

Exercise 69.

Show that the sequence $\{a_n\}$, *where*

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

is convergent.

Solution

$a_{n+1} - a_n = \frac{1}{2(n+1)(2n+1)} > 0$ for all n.

So, the sequence $\{a_n\}$ is nondecreasing.

Also $0 < a_n < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1$, which implies that $\{a_n\}$ is bounded.

Thus $\{a_n\}$ converges.

Exercise 70.

Show that the sequence $\{a_n\}$, *where*

$$a_n = \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

is convergent.

Solution

$a_{n+1} - a_n = \frac{1}{(n+1)!} > 0$ for all n.

So, the sequence $\{a_n\}$ is nondecreasing.

Also $0 < a_n < 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}} < 2$, which implies that $\{a_n\}$ is bounded.

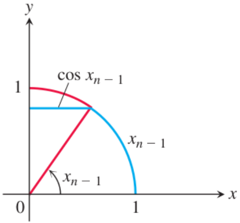
Thus $\{a_n\}$ converges.

A recursive definition of $\pi/2$

If we start with $x_1 = 1$ and define the subsequent terms of $\{x_n\}$ by the rule

$$x_n = x_{n-1} + \cos x_{n-1},$$

we generate a sequence that converges to $\pi/2$.



Subsequences

If the terms of one sequence appear in another sequence **in their given order**, we call the first sequence a **subsequence** of the second.

From a sequence $\{a_n\}$, if we pick up only the terms whose suffices are $n_1, n_2, n_3 \dots$ and generate a new sequence, namely, $\{a_{n_1}, a_{n_2}, a_{n_3}, \dots\}$ mainitaining the same order as in $\{a_n\}$, then this new sequence $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

Definition 71.

Let $\{a_n\}$ be a given sequence. If $\{n_k\}$ is a strictly increasing sequence of natural numbers (i.e., $n_1 < n_2 < n_3 < \cdots$), then $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

Example 72.

1. *The sequences* $\{2, 4, 6, \dots\}$, $\{1, 3, 5, \dots\}$, $\{1, 4, 9, 16, \dots\}$ *are some subsequences of* $\{n\}$.
2. *The sequences* $\{1, \frac{1}{3^3}, \frac{1}{3^5}, \dots\}$, $\{\frac{1}{3^2}, \frac{1}{3^4}, \frac{1}{3^6}, \dots\}$ *are some subsequences of* $\{\frac{1}{3^n}\}$.
3. *The sequence* $\{2^n\}$ *is a subsequence of* $\{n\}$, *whereas the sequence* $\{4, 3, 10, 11, 12, \dots\}$ *is not a subsequence of* $\{n\}$.

Properties of Subsequences

1. The terms of a subsequence occur in the same order in which they occur in the original sequence.
2. Every sequence is a subsequence of itself.
3. The sequence $\{a_{2k}\}$ formed by taking the terms of even index from $\{a_n\}$ is called the **even subsequence** of $\{a_n\}$ and the sequence $\{a_{2k+1}\}$ formed by taking the terms of odd index from $\{a_n\}$ is called the **odd subsequence** of $\{a_n\}$.
4. If $\{u_n\}$ is a subsequence of $\{a_n\}$ and $\{v_n\}$ is a subsequence of $\{u_n\}$, then $\{v_n\}$ is also a subsequence of $\{a_n\}$.
5. The interval between various terms of a subsequence need not be regular.
6. Given a term a_m of the sequence $\{a_n\}$, there is a term of a subsequence following it.

Theorem 73.

If a sequence $\{a_n\}$ converges to A, then every subsequence of $\{a_n\}$ converges to A.

Proof : Let $\{a_n\}$ converge to A and $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.
Claim: $\{a_{n_k}\}$ converges to A.
That is, **to prove that** given any $\varepsilon > 0$, there exists a positive integer N such that

$$|a_{n_k} - A| < \varepsilon, \ \forall n_k \geq N.$$

Let $\varepsilon > 0$ be given. Then, since $a_n \to A$ as $n \to \infty$, corresponding to this ε there exists $N \in \mathbb{N}$ such that

$$|a_n - A| < \varepsilon, \ \forall n \geq N. \tag{3}$$

Choose $n_{k_0} \in \mathbb{N}$ such that $n_{k_0} \geq N$ and $a_{n_{k_0}}$ is an element of $\{a_{n_k}\}$.
Then, for any $n_k \geq n_{k_0}$, we have $n_k \geq n_{k_0} \geq N \implies n_k \geq N$.
Therefore, from (3) we have,

$$|a_{n_k} - A| < \varepsilon, \ \forall n_k \geq N.$$

Since ε is arbitrary, it follows by definition that $a_{n_k} \to A$, as $n_k \to \infty$.

Properties of Subsequences

1. The converse of Theorem (73) is not true. That is, if a subsequence or even infinitely many subsequences of a given sequence converge, then it does not mean that the original sequence should converge. For example, the sequence $\{(-1)^n\}$ does not converge. But, its odd and even subsequences converge to -1 and 1 respectively.
2. However, if all subsequences of a given sequence $\{a_n\}$ converge to the same limit A, only then $\{a_n\}$ converges to A.
3. Based on the above points, to prove that a given sequence is not convergent, it is sufficient to show that two of its subseqeunces converge to different limits.
4. Every sequence contains a monotonic (increasing / decreasing) subsequence.

Exercise 74.

Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

Solution : Let $k(n)$ and $p(n)$ be two order-preserving functions whose domains are the sets of positive integers and whose range are subsets of the positive integers.

Consider two subsequences $a_{k(n)}$ and $a_{p(n)}$ converging to L_1 and L_2 respectively. So

$$|a_{k(n)} - a_{p(n)}| \to |L_1 - L_2| > 0.$$

Hence, there is no positive integer N such that for all $n, m > N$ such that $|a_n - a_m| < \varepsilon$. Thus $\{a_n\}$ is not a Cauchy sequence.

Since every convergent sequence is a Cauchy sequence, so $\{a_n\}$ is not a convergent sequence, thus it diverges.

The following result says that to prove that a given sequence is convergent, it is sufficient to show that its odd and even subsequences converge to the same limit.

Exercise 75.

For a sequence $\{a_n\}$, the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1}. Prove that if $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.

Solution : Let $\varepsilon > 0$ be given. As $a_{2k} \to L$ and $a_{2k+1} \to L$, there are positive integers N_1 and N_2 such that

$$|a_{2k} - L| < \varepsilon \quad \text{for all } 2k > N_1$$

and

$$|a_{2k+1} - L| < \varepsilon \quad \text{for all } 2k + 1 > N_2.$$

Then for any $n > \max\{N_1, N_2\}$,

$$|a_n - L| < \varepsilon \quad \text{whether } n \text{ is even or odd.}$$

Hence $a_n \to L$.

Subsequences of Divergent Sequences

1. $\{a_n\}$ diverges to $+\infty$ if and only if every subsequence of $\{a_n\}$ diverges to $+\infty$ if and only if both the odd and even subsequences of $\{a_n\}$ diverge to $+\infty$.
2. $\{a_n\}$ diverges to $-\infty$ if and only if every subsequence of $\{a_n\}$ diverges to $-\infty$ if and only if both the odd and even subsequences of $\{a_n\}$ diverge to $-\infty$.
3. Knowing that some subsequence of $\{a_n\}$ diverges to $+\infty$ (or $-\infty$) does not necessarily imply $\{a_n\}$ diverges to $+\infty$ (or $-\infty$).
   For example, for the sequence $\{a_n = n(-1)^n\}$, the subsequence $\{a_{2n}\}$ diverges to $+\infty$ whereas $\{a_{2n-1}\}$ diverges to $-\infty$. But, $\{a_n\}$ neither diverges to $+\infty$ nor to $-\infty$.

Exercise 76.

1. *If sequence* $\{a_n\}$ *converges to* A*, then* $\{|a_n|\}$ *converges to* $|A|$.
2. *If a sequence* $\{a_n\}$ *converges to* 0 *and* $\{b_n\}$ *is a bounded sequence, then* $\{a_n b_n\}$ *converges to* 0.
3. $\{a_n\}$ *converges to* 0 *if and only if* $\{|a_n|\}$ *converges to* 0.
4. *If* $a_n \to A$ *and* $a_n \geq k$ *for all* n*, where* k *is any constant, then* $A \geq k$.
5. *If* $a_n \to A$ *and* $a_n \leq k$ *for all* n*, where* k *is any constant, then* $A \leq k$.
6. *If* $a_n \to A$*,* $b_n \to B$ *and* $a_n \leq b_n$*, for all* n*, then* $A \leq B$.
7. *If sequence* $\{a_n\}$ *diverges to* $+\infty$*, then* $\{a_n\}$ *is bounded below, but unbounded above. [But the converse is not true. That is, a sequence which is bounded below and unbounded above need not diverge to* $+\infty$*.]*
8. *If sequence* $\{a_n\}$ *diverges to* $-\infty$*, then* $\{a_n\}$ *is bounded above, but unbounded below. [But the converse is not true. That is, a sequence which is bounded above and unbounded below need not diverge to* $-\infty$*.]*

Extra Problems

Difficult Level - High

Applications of Sequences

According to a front-page article in the December 15, 1992, issue of the Wall Street Journal, Ford Motor Company used about $7\frac{1}{4}$ hours of labor to produce stampings for the average vehicle, down from an estimated 15 hours in 1980. The Japanese needed only about $3\frac{1}{2}$ hours.

Ford's improvement since 1980 represents an average decrease of 6% per year. If that rate continues, then n years from 1992 Ford will use about $S_n = 7.25(0.94)^n$ hours of labor produce stampings for the average vehicle. Assuming that the Japanese continue to spend $3\frac{1}{2}$ hours per vehicle, how many more years will it take Ford to catch up? Find out two ways:

(a) Find the first term of the sequence $\{S_n\}$ that is less than or equal to 3.5.

(b) Graph $f(x) = 7.25(0.94)^x$ and use Trace to find where the graph crosses the line $y = 3.5$.

Applications of Sequences - Compound Interest, Deposits, and Withdrawals

If you invest an amount of money A_0 at a fixed annual interest rate r compounded m times per year, and if the constant amount b is added to the account at the end of each compounding period (or taken from the account if $b < 0$), then the amount you have after $n+1$ compounding period is

$$A_{n+1} = \left(1 + \frac{r}{m}\right) A_n + b. \tag{4}$$

(a) If $A_0 = 1000$, $r = 0.02015$, $m = 12$, and $b = 50$, calculate and plot the first 100 points (n, A_n), How much money is in your account at the end of 5 years? Does $\{A_n\}$ converge? Is $\{A_n\}$ bounded?

(b) Repeat part (a) with $A_n = 5000$, $r = 0.0589$, $m = 12$, and $b = -50$.

(c) If you invest 5000 dollars in certificate of deposite (CD) that pays 4.5% annually, compounded quarterly, and you make no further investments in the CD, approximately how many years will it take before you have 20,000 dollars? What if the CD earns 6.25%

(d) It can be shown that for any $k \geq 0$, the sequence defined recursively by Equation (4) satisfies the relation

$$A_k = \left(1 + \frac{r}{m}\right)^k \left(A_0 + \frac{mb}{r}\right) - \frac{mb}{r}. \tag{5}$$

For the values of the constants A_0, r, m, and b given in part (a), validate this assertion by comparing the values of the first 50 terms of both sequences. Then show by direct substitution that the terms in Equation (5) satisfy the recursion formula in Equation (4).

Applications of Sequences

The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.

(a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and $b - a$.

(b) Show that $p_{n+1} < (b/a)p_n$.

(c) Use part (b) to show that if $a > b$, then $\lim_{n\to\infty} p_n = 0$; in other words, the population dies out.

(d) Now assume that $a < b$. Show that if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Show also that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Deduce that if $a < b$, then

$$\lim_{n\to\infty} p_n = b - a.$$

Applications of Sequences - Logistic Difference Equation

The recursive relation

$$a_{n+1} = ra_n(1 - a_n)$$

is called the logistic difference equation, and when the initial value a_0 is given, the equation defines the logistic sequences $\{a_n\}$. Throughout this exercise we chose a_0 in the interval $0 < a_0 < 1$, say $a_0 = 0.3$.

(a) Choose $r = 3/4$. Calculate and plot the points (n, a_n) for the first 100 terms in the sequence. Does it appear to converge? What do you guess is the limit? Does the limit seem to depend on your choice of a_0?

(b) Choose several values of r in the interval $1 < r < 3$ and repeat the procedures in part(a). Be sure to choose some points near the endpoints of the interval. Describe the behavior of the sequences you observe in your plots.

(c) Now examine the behavior of the sequence for values of r near the endpoints of the interval $3 < r < 3.45$ The transition value $r = 3$ is called a **bifurcation value** and the new behavior of the sequence in the interval is called an **attracting 2-cycle.** Explain why this reasonably describes the behavior.

(d) Next explore the behavior for r values near the endpoints of each of the intervals $3.45 < r < 3.54$ and $3.54 < r < 3.55$, Plot the first 200 terms of the sequences. Describe in your own words the behavior observed in your plots for each interval. Among how may values does the sequence appear to oscillate for c interval? The values $r = 3.45$ and $r = 3.54$ (rounded to two decimal places) are also called bifurcation values because the behavior of the sequences changes as r crosses over those values.

(e) The situation gets even more interesting. There is actually an increasing sequence of bifurcation values $3 < 3.45 < 3.54 < \cdots < c_n < c_{n+1} \cdots$ such that for $c_n < r < c_{n+1}$ the logistic sequence $\{a_n\}$ eventually oscillates steadily among 2^n values, called an **attracting 2^n-cycle.** Moreover, the bifurcation sequence $\{c_n\}$ is bounded from above by 3.57 (so it converges). If you choose a value $r < 3.57$ you will observe a 2^n-cycle of some sort. Choose $r = 3.5695$ and plot 300 points.

(f) Let us see what happens when $r = 3.57$. Choose $r = 3.65$ and calculate and plot the first 300 terms of $\{a_n\}$. Observe how the terms wander around in an unpredictable, chaotic fashion. You cannot predict the value of a_{n+1} from previous values of the sequence.

(g) For $r = 3.65$ choose two starting values of a_n that are close together, say, $a_0 = 0.3$ and $a_0 = 0.301$. Calculate and plot the first 300 values of the sequences determined by each starting value. Compare the behaviors observed in your plots. How far out do you go before the corresponding terms of our two sequences appear to depart from each other? Repeat the exploration for $r = 3.75$. Can you see how the plots look different depending on your choice of a_0? We say that the logistic sequence is sensitive to the initial condition a_0.

1. **Bolzano-Weierstrass theorem** : Every bounded sequence of real numbers has a convergent subsequence.
2. The set of limit points of a bounded sequence has the greatest and the least members.
3. Every bounded sequence with a unique limit is convergent.
4. A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.
5. If $a > 0$, and p is real, then $\lim_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$.
6. If $\{a_n\}$ converges and $\{b_n\}$ diverges, show that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\{a_n + b_n\}$ is divergent.
7. Given that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $\{S_n\}$ and $\{T_n\}$ are two sequences, where $S_n = \max\{a_n, b_n\}$, $T_n = \min\{a_n, b_n\}$. Show that the sequences $\{S_n\}$ and $\{T_n\}$ are convergent and that $\lim_{n\to\infty} S_n = \max\{a, b\}$ and $\lim_{n\to\infty} T_n = \min\{a, b\}$.

Exercises

1. **Cauchy's First Theorem on Limits :**
   Show that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \left(\frac{a_1 + a_2 + \cdots + a_n}{n} \right) = L$.
2. Using Cauchy's first theorem on limits, prove the following :
   (a) If a sequence $\{a_n\}$ of positive terms converges to a positive limit L, then so does the sequence $\left\{ (a_1 a_2 \ldots a_n)^{1/n} \right\}$ of its geometric terms.
   (b) If $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, where $|L| < 1$, then $\lim_{n\to\infty} a_n = 0$.
   (c) If $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, where $L > 1$, then $\lim_{n\to\infty} a_n = \infty$.
3. Show that $\lim_{n\to\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n = 0, |x| < 1$.
4. We proved that a sequence $\{a_n\}$ converges to L if and only if every subsequence converges to L. Show that $\lim_{n\to\infty} a_n = \infty$ $(-\infty)$ if and only if every subsequence of $\{a_n\}$ tends to ∞ $(-\infty)$.

1. If $\{a_n\}$ is a sequence such that

$$a_{n+1} = \sqrt{\frac{ab^2 + a_n^2}{a+1}}, \quad b > 0 \quad \text{for all } n$$

and $a_1 = a > 0$, then show that the sequence $\{a_n\}$ converges to b.

2. If $\{a_n\}$ is a sequence of positive real numebrs such that $a_n = \frac{a_{n-1}+a_{n-2}}{2}$, for all $n \geq 2$, then show that $\{a_n\}$ converges. Find $\lim\limits_{n\to\infty} a_n$.

3. Show that the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right), \quad \text{for all } n \geq 1$$

and $a_1 = 0$ converges to 3.

Exercises

1. If a sequence $\{a_n\}$ is defined by $a_n = \frac{b}{1+a_{n-1}}$, where $b > 0, a_1 > 0, n \geq 2$, then show that the sequence converges to the positive root of the equation
$$x^2 + x - b = 0.$$
2. Two sequence $\{a_n\}$ and $\{b_n\}$ are defined inductively by $a_1 = \frac{1}{2}$ and $b_1 = 1$ and
$$a_n = \sqrt{a_{n-1} b_{n-1}} \quad \text{and} \quad \frac{1}{b_n} = \frac{1}{2}\left[\frac{1}{a_n} + \frac{1}{b_{n-1}}\right] \quad n = 2, 3, 4, \ldots.$$
Prove that $a_{n-1} < a_n < b_n < b_{n-1},\ n = 2, 3, 4, \ldots$ and deduce that both the sequences converge to the same limit L, where $\frac{1}{2} < L < 1$.

Extra Problems

Difficult Level - Average

Exercises

Find a formula for the general term a_n (the nth term) of the sequence

1. $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5}, \ldots.$
2. $0.9, 0.99, 0.999, 0.9999, \ldots.$
3. $\frac{1}{2}, \frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}, \ldots.$
4. $\left\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\right\}$

assuming that the pattern of the first few terms continues.

Exercises

1. The limit of the sequence $\left\{ \ln(4(n - \sqrt{n^2 - n})) \right\}$ is ______.
2. Show that the sequence defined by

$$a_1 = 2, \quad a_{n+1} = \frac{1}{3 - a_n}$$

satifies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find it its limit.

3. Consider two sequences $\{a_n\}$ and $\{b_n\}$ given by $a_n = (1/n)^{(\sin(1/n))}$ and $b_n = (\sin(1/n))^{(1/n)}$. Choose the correct answer.

(a) $a_n \to 0, b_n \to 0$
(b) $a_n \to 0, b_n \to 1$
(c) $a_n \to 1, b_n \to 1$
(d) $a_n \to 1, b_n \to 0$

1. Let $a_1 = 1$ and $a_{n+1} = \frac{a_n}{2} + \frac{2}{a_n},\ n \geq 1$. Find the limit of the sequence $\{a_n\}$ if it exists.
2. Let p be a prime number. For each k, $0 \leq k \leq p-1$, the sequence $\{a_{np+k}\}$ converges to 2 as $n \to \infty$. Show that $\{a_n\}$ converges to 2.
3. Let c and a_1 be two positive real numbers. Consider the sequence $\{a_n\}$ defined by

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$$

for all $n \in \mathbb{N}$. Show that $a_{n+1} \leq a_n \ \forall\ n \geq 2$ and $\{a_n\}$ is bounded from below. Find the limit of the sequence $\{a_n\}$.

Exercises

1. Find the limit of the following sequences whose nth term is given by the formula
   (i) $\dfrac{(-1)^n}{n+1}$ (ii) $\dfrac{2n}{3n^2+1}$ (iii) $\dfrac{2n^2+3}{3n^2+1}$
   (**Ans**: (i) 0, (ii) 0, (iii) 2/3).
2. Show that the sequences given above converge to the corresponding limits by $\varepsilon - N$ definition.
3. Discuss the convergence of the sequence (a_n) defined recursively by (i) $a_1 = 1,\ a_{n+1} = 2 - 3a_n,\ n = 1, 2, \ldots$ (ii) $a_1 = 1$ and $a_{n+1} = \frac{a_n}{1+a_n},\ n = 1, 2, \ldots$.
   (**Ans**: (i) divergent (ii) convergent)
4. Let $a_1 = 2,\ a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}),\ n = 1, 2, \ldots$ Show that $\{a_n\}$ is decreasing and bounded below by $\sqrt{2}$.

Exercises

1. Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}.$$

   Ans: 2.

2. Find the limit of (i) $a_n = \left(1 + \frac{1}{n}\right)^n$ (ii) $a_n = \left(\frac{3n+1}{3n-1}\right)^n$.

   (**Ans**: (i) e, (ii) $e^{2/3}$).

3. For any real number x, show that $\left\{\frac{x^n}{n!}\right\}$ converges.

4. Show that $\left(\frac{\log n}{n^c}\right) \to 0$ for any $c > 0$.

Exercises

1. Give an example of a continuous function $f(x)$ and a sequence (a_n) such that $f(a_n)$ converges but (a_n) diverges.
2. Discuss the convergence of
(i) $\frac{\sin^2 n}{2n}$ (ii) $\frac{n!}{2^n 3^n}$ (iii) $\frac{n!}{n^n}$ (iv) $n^{1/n}$ (v) $\sqrt{n} - \sqrt{n+1}$.
(**Ans**: (ii) divergent. (i),(ii),(iv),(v) convergent.)
3. Give an example of a sequence (a_n) of positive numbers which converges but the sequence (b_n) diverges where $b_n = \frac{a_{n+1}}{a_n}$.
4. Let $a_1 = a, a_2 = f(a_1), a_3 = f(a_2) = f(f(a)), \ldots, a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.

1. Define the sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$0 < b_1 < a_1, a_{n+1} = \frac{a_n + b_n}{2} \text{ and } b_{n+1} = \sqrt{a_n b_n} \text{ for } n \in \mathbb{N}.$$

   Show that $\{a_n\}$ and $\{b_n\}$ both tend to the same limit. This limit is called the arithmetic-geometric mean of a_1 and b_1.
2. Let the sequence $\{a_n\}$ be defined by $a_n = \lim_{n\to\infty} \frac{[x]+[2x]+\ldots+[nx]}{n^2}$, where x is a real number. Is this sequence convergent? If so, what is the limit? (**Ans**: $x/2$)
3. Show that the sequence $\{(1 + 1/n)^n\}$ is a monotone increasing sequence, bounded above.

Exercises

1. Let $\{b_n\}$ be a bounded sequence which satisfies the condition $b_{n+1} \geq b_n - \frac{1}{2^n}, n \in \mathbb{N}$. Show that the sequence $\{b_n\}$ is convergent.
2. For $c > 2$, the sequence $\{p_n\}$ is defined recursively by $p_1 = c^2, p_{n+1} = (p_n - c)^2, n > 1$. Show that the sequence (p_n) strictly increases.
   [Hint : By induction, first prove that $p_n > 2c$.]

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